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A

TREATISE

ON

ARITHMETIC,

COMBINING

ANALYSIS AND SYNTHESIS,

ADAPTED TO

THE BEST MODE OF INSTRUCTION IN COMMON SCHOOLS AND ACADEMIES.

BY

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PREFACE.

In an experience of many years in teaching Arithmetic, the author of the following treatise has thought that, with many excellences, there were also many defects in the best books in use. To correct these defects and to multiply the excellences has been his constant aim. This is his only apology for presenting a other school-book in a department already overburdened.

It has been the guiding principle to be clear, brief, accurate, logical. Subjects are arranged, first, with reference to their dependence, and, secondly, with reference to their importance and simplicity — the less difficult and more practical first, and the more intricate and less important afterwards.

In Reduction, those examples requiring a familiar acquaintance with fractions have been deferred until fractions have been discussed; and in Fractions, the several operations have been arranged with strict regard to the dependence of principles, contrary to the almost uniform arrangement of other works.

Some articles of a practical business nature, not usually found in arithmetics, have been introduced, and special care has been taken to adapt the work to the wants of the business community; yet the science of numbers has not been forgotten, but the definitions and manner of discussion have been designed to prepare the pupil to enter upon the study of Algebra with pleasure and profit.

It has been assumed that the learner has some knowledge of the properties and relations of numbers (and no scholar should be allowed to study *Written* Arithmetic until he is familiar with the modes of reasoning in *Intellectual* Arithmetic), yet it is believed that every intricate principle has been clearly presented before its aid is required for the solution of an example.

It has been designed to give answers enough to inspire confidence in the learner, and yet to omit enough to secure the discipline resulting from proving the operations.

At the close of the work, an extended Supplement has been added, designed to be suggestive (and this has been a leading idea in the whole work) rather than full or consecutive, it being considered the great end of school discipline to lead the pupil to *think for himself.*

In submitting this treatise to the intelligence and candid criticism of school committees and practical teachers, the author takes pleasure in acknowledging deep indebtedness to the Principal, the Associate Teachers and Treasurer of Phillips Academy, and to other eminent teachers and men of practical knowledge, for important suggestions and valuable criticisms in the preparation of the work.

A Key, containing the Answers not given in this book, is published for the use of Teachers.

PHILLIPS ACADEMY, ANDOVER,
July 15, 1857.

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SUPPLEMENT.

ARITHMETIC.

Article **1**. Arithmetic* is the *science* of numbers and the *art* of computation.

(a) A number is a *unit* or a *collection of units* — a unit† being the least whole number, viz. *one*.

2. Arithmetic employs *six* different operations, viz. *Notation*, *Numeration*, *Addition*, *Subtraction*, *Multiplication* and *Division*.

Note. — These operations are variously combined, giving rise to a great number of rules.

§ 1. NOTATION.

3. Notation is the art of expressing numbers and their relations to each other by means of *figures and other symbols*.

4. To express numbers, the ten Arabic figures or *digits*,‡

0,	1,	2,	3,	4,	5,	6,	7,	8,	9,
naught,	one,	two,	three,	four,	five,	six,	seven,	eight,	nine,

are in general use.

* *Arithmetic* is from the Greek Ἀριθμητική (sc. τέχνη), *the art or science of numbers*.

† *Unit*, from the Latin *unus*, which means *one*.

‡ *Digit*, from the Latin *digitus*, *a finger*; a term probably applied to figures from the custom of *counting upon the fingers*.

5. The seven Roman letters,

I,	V,	X,	L,	C,	D,	M,
one,	five,	ten,	fifty,	one hundred,	five hundred,	one thousand,

are sometimes used to express numbers.

6. The sign of *dollars* is written thus, $; e. g. $2 represents *two dollars*; $10, *ten dollars*, etc.

7. The sign of *equality*, $=$, signifies that the quantities between which it stands, are equal to each other; thus, $1 $=$ 100 cents, i. e. one dollar equals one hundred cents.

NOTE.—An expression in which the sign of equality occurs, is called *an equation*. That portion of the equation which precedes the sign of equality is the *first member*, and that which follows, the *second member* of the equation.

8. The sign of *addition*, $+$, called *plus*, denotes that the quantities between which it stands are to be added together; thus, $3 + 2 = 5$, i. e. three plus two equals five, or three and two are five.

9. The sign of *subtraction*, $-$, called *minus*, signifies that the number after it is to be taken from the number before it; thus, $7 - 4 = 3$, i. e. seven minus four, or seven diminished by four, equals three.

10. The sign of *multiplication*, $\times$, signifies that the two numbers between which it stands are to be multiplied together; thus, $6 \times 5 = 30$, i. e. six multiplied by five equals thirty; or more familiarly, six times five are thirty.

11. The sign of *division*, $\div$, indicates that the number before it is to be divided by the number after it; thus, $8 \div 2 = 4$, i. e. eight divided by two equals four.

(a) Division is also frequently indicated thus, $\frac{8}{2} = 4$; also by two dots, thus, $8 : 2 = 4$; i. e. eight divided by two equals four.

12. Three dots, $\therefore$, are the symbol for *therefore*; e. g. $8 \div 2 = 4$ and $\frac{8}{2} = 4$, $\therefore 8 \div 2 = \frac{8}{2}$, i. e. *therefore* these two forms $8 \div 2$ and $\frac{8}{2}$, have the same signification.

NOTE.—Other signs and their signification will be given when their aid is needed.

§ 2. NUMERATION.

13. NUMERATION is the art of reading numbers which have been expressed by figures.

14. The first Arabic figure, 0, is called *a cipher, naught* or *zero,* and, standing alone, signifies *nothing.* Each of the remaining nine figures,

1,	2,	3,	4,	5,	6,	7,	8,	9,
one,	two,	three,	four,	five,	six,	seven,	eight,	nine,

represents the number placed under it, and, for convenience in distinguishing them from 0, they are called *significant figures.*

NOTE. — The terms *significant* and *insignificant* are used *technically;* 0 is as really significant as any other figure.

15. Each *significant* figure has *two values;* one of which is *constant,* (i. e. *always the same,*) the other, *variable;* thus, in each of the numbers 2, 20 and 200, the left-hand figure is *two;* but in the *first* it is two *units;* in the *second,* two *tens;* and in the *third,* two *hundreds.*

The former of these values is the *inherent* or *simple* value, and the latter is the *local* or *place* value.

16. The *value* of a figure is made *ten fold* by removing it *one* place towards the *left; a hundred fold* by removing it *two* places, etc.

17. For convenience in reading, the figures of large numbers are frequently separated by commas into periods or groups.

18. There are two methods of grouping — the *French* and the *English.* By the French method a period consists of *three* figures — by the English, of *six.* The French method is most convenient and principally used in this country.

19. By the French method the first or right-hand period contains units, tens and hundreds and is called the *period of units*, the second period contains thousands, tens of thousands and hundreds of thousands and is called the *period of thousands;* etc. as in the following

FRENCH NUMERATION TABLE.

Quintillions.	Hundreds of Quadrillions,	Tens of Quadrillions,	Quadrillions,	Hundreds of Trillions,	Tens of Trillions,	Trillions,	Hundreds of Billions,	Tens of Billions,	Billions,	Hundreds of Millions,	Tens of Millions,	Millions,	Hundreds of Thousands,	Tens of Thousands,	Thousands,	Hundreds,	Tens,	Units.
8,	7	6	9,	5	4	0,	7	0	6,	4	7	6,	0	0	1,	8	4	3
7th period, Quintillions.	6th period, Quadrillions,			5th period, Trillions,			4th period, Billions,			3d period, Millions,			2d period, Thousands,			1st period, Units,		

20. The value of the figures in this table, expressed in words, is eight quintillions, seven hundred and sixty-nine quadrillions, five hundred and forty trillions, seven hundred and six billions, four hundred and seventy-six millions, one thousand, eight hundred and forty-three.

21. The table can be extended to any number of places, adopting a new name for each succeeding period. The periods above quintillions, are sextillions, septillions, octillions, nonillions, decillions, undecillions, duodecillions, etc.

Exercises in Numeration by the French Method.

22. Let the learner read the following numbers : —

1.	17	7.	23,486,927
2.	156	8.	74,600,007,468
3.	9,874	9.	9,999,999,999,999
4.	76,523	10.	471,654,769,853,670
5.	890,204	11.	5,476,906,757,000,000
6.	4,076,470	12.	14,000,000,456,447,993

23. By the English method the first period contains units, tens, hundreds, thousands, tens of thousands and hundreds of thousands and is called the *period of units;* the second period contains millions, tens of millions, hundreds of millions, thou-

sands of millions, tens of thousands of millions and hundreds of thousands of millions and is called the *period of millions;* etc., as in the following

ENGLISH NUMERATION TABLE.

Tens of Trillions.	4	4th period, Trillions.
Trillions,	3,	
Hundreds of Thousands of Billions,	7	3d period, Billions,
Tens of Thousands of Billions,	9	
Thousands of Billions,	8	
Hundreds of Billions,	6	
Tens of Billions,	5	
Billions,	4,	
Hundreds of Thousands of Millions,	8	2d period, Millions,
Tens of Thousands of Millions,	7	
Thousands of Millions,	4	
Hundreds of Millions,	6	
Tens of Millions,	7	
Millions,	8,	
Hundreds of Thousands,	9	1st period, Units,
Tens of Thousands,	6	
Thousands,	4	
Hundreds,	0	
Tens,	2	
Units,	8.	

24. The value of the figures in this table, is forty-three trillions; seven hundred ninety-eight thousand, six hundred and fifty-four billions; eight hundred seventy-four thousand, six hundred and seventy-eight millions; nine hundred sixty-four thousand and twenty-eight.

25. The *names* of the figures *and their values are the same* in the two tables for the first nine places from the right, after which they are *alike in value but different in name.* A trillion by the English method is much more than by the French.

EXERCISES IN NUMERATION BY THE ENGLISH METHOD.

26. Read the following numbers : —

1. 87,658746,987694,127346
2. 93467,865493,217684,729111
3. 47,608000,000047,699998,743270
4. 777,777777,777777,777777,777777
5. 9000,000000,000000,000000,000000

Exercises in Notation.

27. Let the learner express the following numbers in figures by the French method: —

1. One thousand, three hundred and fourteen.
2. Seventeen millions and thirty-six thousand.
3. Sixty-five trillions, four millions and six.
4. One hundred and fifty-three decillions, one hundred and forty-nine octillions, nine hundred and ninety-nine quintillions, forty-eight quadrillions, seven hundred and forty-seven thousand, nine hundred and ninety-nine.

28. Express the following numbers in figures by the English notation: —

1. Fourteen billions; three hundred fifty-six thousand, two hundred and fifty-seven millions; five hundred and twenty-five thousand, seven hundred and forty-one.
2. Two quintillions; five thousand quadrillions; two hundred forty-two thousand, seven hundred and fifty-two millions; two hundred and fourteen.

Note. — These and other exercises will be varied and extended by the teacher as circumstances may dictate.

29. TABLE OF ROMAN NUMERALS.

I	1	XXI	21
II	2	XXIV	24
III	3	XXV	25
IV	4	XXIX	29
V	5	XXX	30
VI	6	XL	40
VII	7	L	50
VIII	8	LX	60
IX	9	XC	90
X	10	C	100
XI	11	CCCC	400
XII	12	D	500
XIII	13	DCCCC	900
XIV	14	M	1000
XV	15	MD	1500
XVI	16	MDC	1600
XVII	17	MDCLXV	1665
XVIII	18	MDCCXLIX	1749
XIX	19	MDCCCXVI	1816
XX	20	MDCCCLVII	1857

(a) When two or more letters of equal value are united, or when a letter of less value follows one of greater, the *sum* of their values is indicated; thus, XXX = 30, LXV = 65, CC = 200, etc.

(b) When a letter of less value precedes one of greater, the *difference* of their values is indicated; as, IX = 9, XL = 40 etc.

(c) When a letter of less value stands between two of greater value, the less is to be taken from that which follows it and the remainder to be added to that which precedes it; as, XIV = 14, CXL = 140, etc.

Exercises in Roman Notation.

30. Express the following numbers by letters: —

1. Twelve. Ans. XII.
2. Eighteen. Ans.
3. Twenty-nine.
4. Ninety-nine.
5. Two hundred and eighty-four.
6. One thousand, four hundred and forty-six.
7. One thousand, six hundred and forty-four.
8. The present year, A. D. ——.

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§ 3. ADDITION.

31. Addition is the putting together of two or more numbers of the same kind, to find their *sum* or *amount*.

32. This may be done by the following

Rule. — *Write the numbers in order, units under units, tens under tens, etc. Draw a line beneath, add together the figures in the units' column and, if the sum be less than ten, set it under that column; but, if the sum be ten or more, write the units as before and add the tens to the next column. Thus proceed till all the columns are added.*

33. Ex. 1. Add together 27, 93 and 145.

OPERATION.
27
93
145
265

Having arranged the numbers, we add the column of units; thus, 5 and 3 are 8, and 7 are 15 units (= 1 ten and 5 units). The 5 units are placed under the column of units and the 1 ten is added to the column of tens; thus, 1 and 4 are 5, and 9 are 14, and 2 are 16 tens (= 1 hundred and 6 tens). The 6 tens are set under the tens and the 1 hundred is added to the 1 hundred in the third column, making 2 hundreds to be set under the third column.

	2.	3.	4.	5.
	469	879	5632	98423
	874	458	4561	78621
	324	376	7894	85432
Ans.	1667	1713	18087	262476

	6.	7.	8.	9.
	469	3579	123456	3607842
	874	6842	789012	24681
	327	6070	345678	246
	984	8479	901234	874690
	387	5164	655360	390625
	625	1024	20736	21025
	676	9801	412090	960400
	729	3721	768241	1367631
Ans.				

34. PROOF. — *Having added several numbers together, it is desirable to* TEST *the* ACCURACY *of the work; this* TEST *is called the* PROOF *and is applied in several different ways. The usual mode is to begin at the* TOP *of the units' column and add* DOWNWARD. IF THE WORK IS RIGHT THE TWO SUMS WILL BE ALIKE. *By this process we* COMBINE *the figures* DIFFERENTLY, *and* HENCE *shall* PROBABLY *detect any mistake which may have been made in adding* UPWARD.

NOTE 1. — The operation *called proof,* in this and the following Articles, *only serves to strengthen the probability* that the work is right.

ILLUSTRATION.

	Ex. 10.
	37684
	48297
	68746
	94852
Sum,	249579
Proof,	249579

In adding *upward* we say, 2 and 6 are 8, and 7 are 15, and 4 are 19, etc.; but, in adding *downward*, we say, 4 and 7 are 11, and 6 are 17, and 2 are 19, etc., thus obtaining the *same result*, but by *different combinations.*

If we do not obtain the same result by the two methods, one operation or the other is wrong — perhaps both — and the work must be *carefully* performed again.

NOTE 2. — *In adding it is* NOT *desirable to* NAME *the figures that we add; thus, in the above example, instead of saying* 2 *and* 6 *are* 8, *and* 7 *are* 15, *and* 4 *are* 19, *it is* SHORTER *and* THEREFORE BETTER *to say* 2, 8, 15, 19; *setting down the* 9, *say*, 1, 6, 10, 19, 27, *etc. etc.*

NOTE 3. — Much of the labor of the practical accountant consists in adding up *long* columns of figures, and the learner should not rest satisfied till he can *readily* sum up such columns with *unerring accuracy.*

Ex. 11.	12.	13.	14.
24864	88695	50000	44444
11608	44347	25000	22222
38020	93040	15000	66666
49132	90000	55555	55555
12883	95000	54445	33333
12677	95004	53333	11111
24764	47804	56667	88888
24914	68704	84769	77777
24900	98764	25231	33333
24878	58798	34372	11111
19864	95490	55628	55000
27414	98695	72869	33888
29914	96564	37131	40404
37208	90825	46872	48484
18692	95695	63128	27272
21778	92237	84279	61616
23321	98153	25721	53535
24321	94373	94876	35353
34314	93017	15124	47210

Ex. 15. Add 7435, 86424, 75, 987 and 14.

16. What is the sum of 9874, 625 and 49687428?

17. Add four hundred and fifty-six; eight thousand, four hundred and seventy-two; fifteen thousand, seven hundred and twenty-one; forty-three millions, seven hundred and thirty-three thousand, eight hundred and fifty-nine; and ten.

18. A owes to B $176, to C $8796, to D $549, to E $27, to F $5, to G $1111, to H $469 and to I $46978; how much does he owe?

19. What was the aggregate population of our country in 1790, 1800, 1810, etc., the population of each state and territory being as in the following table?

	1790.	1800.	1810.	1820.	1830.	1840.	1850.
Ala.				127,901	309,527	590,756	771,623
Ark.				14,273	30,388	97,574	209,897
Cal.							92,597
D. C.		14,093	24,023	33,039	39,834	43,712	51,687
Ct.	238,141	251,002	262,042	275,202	297,675	309,978	370,792
Del.	59,096	64,273	72,674	72,749	76,748	78,085	91,532
Fla.					34,730	54,477	87,445
Ga.	82,548	162,101	252,433	340,987	516,823	691,392	906,185
Ill.			12,282	55,211	157,445	476,183	851,470
Ind.		4,875	24,520	147,178	343,031	685,866	988,416
Iowa						43,112	192,214
Ky.	73,077	220,955	406,511	564,317	687,917	779,828	982,405
La.			76,556	153,407	215,739	352,411	517,762
Maine	96,540	151,719	228,705	298,335	399,455	501,793	583,169
Md.	319,728	341,548	380,546	407,350	447,040	470,019	583,034
Mass.	378,717	423,245	472,040	523,287	610,408	737,699	994,514
Mich.			4,762	8,896	31,639	212,267	397,654
Miss.		8,850	40,352	75,448	136,621	375,651	606,526
Mo.			20,845	66,586	140,455	383,702	682,044
N. H.	141,899	183,762	214,360	244,161	269,328	284,574	317,976
N. J.	184,139	211,949	245,555	277,575	320,823	373,306	489,555
N. Y.	340,120	586,756	959,049	1,372,812	1,918,608	2,428,921	3,097,394
N. C.	393,751	478,103	555,500	638,829	737,987	753,419	869,039
Ohio		45,365	230,760	581,434	937,903	1,519,467	1,980,329
Pa.	434,373	602,361	810,091	1,049,458	1,348,233	1,724,033	2,311,786
R. I.	69,110	69,122	77,031	83,059	97,199	108,830	147,545
S. C.	249,073	345,591	415,115	502,741	581,185	594,398	668,507
Tenn.	35,791	105,602	261,727	422,813	681,904	829,210	1,002,717
Texas							212,592
Vt.	85,416	154,465	217,713	235,764	280,652	291,948	314,120
Va.	748,308	880,200	974,622	1,065,379	1,211,405	1,239,797	1,421,661
Wis.						30,945	305,391
Territ.							92,298
Ans.	3,929,827	5,305,937					23,191,876

20. The population of England in 1851 was 16921888; of Scotland, 2888742; of Wales, 1005721; of Ireland, 6515794. What was the population of Great Britain and Ireland?

21. The area of Maine is 35000 square miles; N. H., 8030; Vt., 8000; Mass., 7250; R. I., 1200; Ct., 4750. What is the area of New England?

22. The area of New England in 1853, was about 64236 square miles; of the Middle States, i. e. N. Y., N. J., Pa. and Del., 101971; of other states north of the Ohio and east of the Mississippi, 239349; south of the Ohio and east of the Mississippi, 442673; west of the Mississippi, 723997; of the Territories, 1734645. What was the area of our country?

Ans. 3306865 sq. m.

23. William the Conqueror began to reign in England in the year 1066 and reigned 21 years; William II reigned 13 years; Henry I, 15 years; Stephen, 39 years; Henry II, 35 years; Richard I, 10 years; John, 17 years; Henry III, 56 years; Edward I, 35 years; Edward II, 20 years; Edward III, 50 years; Richard II, 22 years. In what year was Richard II dethroned?

24. The distances of several places from Washington are as follows, viz. Augusta, Me., 595 miles; Concord, N. H., 481 miles; Montpelier, Vt., 516 miles; Boston, Mass., 440 miles; Providence, R. I., 400 miles; Hartford, Ct., 335 miles. What distance will that man travel who goes from Augusta to Washington, thence to Concord, thence back to Washington and so on until he has visited each place in succession and finally goes from Hartford to Augusta, a distance of 250 miles?

25. Suppose a member of Congress to be elected in each of the places named in Ex. 24, what will be their aggregate travel in going to Washington and returning to their homes once each?

26. A merchant bought 6 bales of cloth measuring 125, 99, 384, 162, 400 and 399 yards, respectively; how many yards did he buy?

27. A butcher bought eight oxen which weighed, after they were dressed, 517, 493, 862, 1127, 419, 768, 1243 and 987

pounds, respectively; how many pounds of beef did he purchase?

28. A, B, C and D, commencing trade together, furnished money as follows — A, \$3465; B, \$2700; C, \$5575; and D, \$6000. What was the total capital?

29. There are five numbers; the 1st is 476; the 3d is 9768; the 2d is the sum of the 1st and 3d; the 5th is the sum of the 1st, 2d and 3d; and the 4th is the sum of the 2d and 5th. What is the sum of the five numbers?

30. The difference of two numbers is 876954 and the smaller is 7869432. What is the larger? and what the sum of the two numbers?

31. The cost of the American army for five successive years, commencing in 1812, was \$12187046, \$19906362, \$20608366, \$15394700 and \$16475412; what was the cost for five years?

32.	33.	34.
5487962	478693252	246987432
276	989742624	523478761
997867985	396872748	532487649
98746958	769879	772496778
347697842	4769427	624783624
999999947	953643468	728637453
369874	8742763	432789645
7689106	427410637	324326728
5879629	274812	674235978
427864278	374444693	472641324
376987423	546873940	264731598
572135722	953201457	416247565
994120371	25731271	214652911
992310894	994012377	564562161
999630126	99993529	456099244
53	521306748	160251436
652302158	10257376	264105265
901253042	603127252	116392111
2132015	999230121	356401240
999999724	995230573	365410128
994512038	46356532	641901457

NOTE 4. — It is customary to separate dollars and cents by a point; thus, $6.82 is read, six dollars and eighty-two cents.

Ex. 35.	36.
$1643.42	$4357.00
347.31	429.66
56.25	3906.25
3.33	6789.75
78.16	1150.00
1832.43	742.83
741.50	5432.10
259.30	212.18
83.33	79.20
16.79	8901.31
7128.23	4728.53
773.19	940.42
940.43	2544.96
59.75	366.03
337.16	267.30
42.58	22.81
18.76	256.00
1530.21	55.02
55.02	268.34
586.75	367.35
142.04	2269.54
34.75	200.41

NOTE 5. — Accountants usually separate long columns of figures into parts by drawing horizontal lines, add the numbers between these lines, set the sums at the right, and then add these sums, as in the annexed example.

Ex. 37.	
$8756.92	
47.85	
432.54	
762.94	
5.48	
14.52	
23.07	10043.32
567.49	
874.32	
56.15	
2.12	
75.15	
56.02	
47.14	1678.39
2.25	
9.15	
87.42	
569.87	
24.56	693.25
842.15	
47.96	
81.15	
46.53	
487.20	
537.00	
478.69	
2468.25	
42.16	
8.05	
.50	5039.64
Ans.	$17454.60

38. The cost of taking and printing the census of the United States in 1790, was $44377.28; in 1800, $66109.04; in 1810, $178444.67; in 1820, $208525.99; in 1830. $378545.13; in 1840, $832370.95; in 1850, $1318027.53. What has been the cost of these seven censuses? Ans. $3026400.59.

39. A farmer owns a farm worth \$4775, a pair of oxen worth \$115, a horse worth \$100, six cows worth \$30 each, and other property worth \$1550; what is the value of his estate?

40. The area of North America in 1853, was about as follows, viz. — United States, 3,306,865 square miles; British America, 3,050,398; Mexico, 1,038,834; Central America, 203,551; Russian America, 394,000; Danish America, 380,000. What is the area of North America?

41. $66942 + 48 + 7432987 + 463 + 87425 =$ how many?
Ans. 7587865.

42. $874259876 + 427 + 895276528 + 4307698742 = ?$

43. $92 + 27 + 56 + 99 + 88 + 77 + 66 + 55 + 44 + 22 + 7 + 33 = ?$

44. \$86.94 + \$17.06 + \$45.64 + \$43.26 + \$72.18 + \$9.82 = ?

45. \$69432.87 + \$469.84 + \$7694.18 + 42693.14 = ?

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§ 4. SUBTRACTION.

35. SUBTRACTION * is taking a *less number* from a *greater* to find their *difference.*

The greater number is called the *minuend;* † the less, *subtrahend;* ‡ the difference, *remainder.*

36. Subtraction is the reverse of addition.

RULE. — 1. *Write the less number under the greater, — units under units, tens under tens, etc. — and draw a line beneath.*

2. *Beginning at the right hand, take each figure of the subtra-*

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* *Subtraction,* from the Latin *subtraho, to draw from under, to take away, to subtract.*

† *Minuend,* from the Latin *minuendus, to be diminished.*

‡ *Subtrahend,* from the Latin *subtrahendus, to be subtracted from.*

lend from the figure above it and set the remainder under the line.

3. *If any figure in the subtrahend is greater than the figure above it, add* TEN *to the upper figure and take the lower figure from the* SUM ; *set down the remainder and* ADD ONE *to the next figure in the subtrahend.*

Ex. 1.

Minuend,	8 4 6
Subtrahend,	4 2 1
Remainder,	4 2 5

This example is solved by the first two sections of the rule and needs no explanation.

Ex. 2.

Minuend,	4 8 3
Subtrahend,	2 5 7
Remainder,	2 2 6

In this example we cannot take 7 units from 3 units, but if *one* of the 8 *tens* is put with the 3 *units* it will make 13 units, and 7 units taken from 13 units will leave 6 units. Now as *one* of the 8 *tens* has been put with the 3 *units*, there will remain but 7 tens and we may take the 5 tens from 7 tens ; or, following the *rule*, we may add *one* ten to the 5 tens and take the *sum* (6 tens) from 8 tens, since the result will be 2 tens by either process.

Ex. 3.

	(5)	(9)	(12)
Minuend,	6	0	2
Subtrahend,	4	3	8
Remainder,	1	6	4

Here we cannot take 8 from 2, nor can we borrow from the tens' place, as that place is occupied by 0 ; but we can borrow *one* of the 6 *hundreds* and separate the one hundred into 9 *tens* and 10 *units ;* then, putting the 9 tens in the place of tens and adding the 10 units to the 2 units, we can subtract 8 from 12, 3 from 9 and 4 from 5.

NOTE. — This process will probably be more readily *understood* by the young learner than that given in the *rule,* though the latter, being thought more convenient, is usually adopted.

37. PROOF. — *Add the subtrahend and remainder together and the sum should be the minuend.*

NOTE. — This proof rests upon the axiom that *the whole of a thing is equal to the sum of all its parts ;* thus, the *minuend* is divided into the *two parts — subtrahend* and *remainder ; hence* the *sum* of those parts *must be* the *minuend*

Ex. 4.

Minuend,	6 8 7 4 5
Subtrahend,	2 6 8 5 4
Remainder,	4 1 8 9 1
Proof,	6 8 7 4 5

As the *sum* of the subtrahend and remainder is the minuend, the work is *probably* right. (Art. 34, Note 1.)

	5.	6.	7.
Minuend,	9 8 7 5	5 3 2 7 6 9	5 7 8 4
Subtrahend,	2 6 5	2 7 8 4 9 3	3 2 9 6
Remainder,	9 6 1 0	2 5 4 2 7 6	
Proof,	9 8 7 5	5 3 2 7 6 9	

	8.	9.
Min.	9 8 5 6 4 3 2 1 8 2 7 6 9 4 2	4 2 7 0 6 8 9 4
Sub.	2 9 4 5 2 7 8 2 4 3 6 7 8 9 5	3 4 8 7 9 6 3 2
Rem.		
Proof,		

10.	11.
7 6 9 8 4 2 0 0 4 6 7 8 4 2 9	5 7 8 4 2 9 6 9 4 3 2 4
3 0 4 6 8 7 5 4 2 9 8 4 6	6 7 9 8 4 3 0 0 0

12. From 6342 take 2735. Ans. 3607.

13. 8394769874 — 2487962893 = how many?

Ans. 5906806981.

14. 8479326948 — 5274679894 = ?

15. 2734698254 — 984273205 = ?

16. 4000082000 — 827400832 = ?

17. 5000000000 — 4999999999 = ?

18. 7069842374 — 7000000000 = ?

19. 876678876 — 678876678 = ?

20. The minuend in a certain example is 4798 and the subtrahend is 2653; what is the remainder?

21. The subtrahend is 576 and the minuend is 9874654; what is the remainder?

22. The minuend is 9009 and the remainder is 7692; what is the subtrahend?

23. What is the difference between 876 and 987487?

24. What is the difference between 7690843254 and 222?

25. Minuend = 874; subtrahend = 269; remainder = ?

26. Minuend = 8746932; remainder = 999; subtrahend = ?

27. Remainder = 4967; minuend = 879694; subtrahend = ?

28. Columbus discovered America A. D. 1492; how many years have since elapsed?

29. The difference between two numbers is 8347; the greater number is 15306. What is the less?

30. What number is that to which if 768 be added the sum will be 987105?

31. What number is that which, taken from 687945, leaves 87640?

32. From seventy-six millions and thirty-two, take fourteen thousand three hundred and seventy-eight.

33. The sum of two numbers is 5769842; the greater number is 4839842. What is the less?

34. The sum of two numbers is 2789; the less is 879. What is the greater?

35. The greater of two numbers is 86942; the less is 6894. What is the difference?

36. A treaty of peace was made with Great Britain in 1783 and war was again declared in 1812; how long did peace continue?

37. Washington was born in 1732 and died in 1799; at what age did he die?

38. The number of states in the American Union at the adoption of the Federal Constitution was 13; the number at the present time, (1857,) is expressed by the same figures taken in the reverse order. How many states have been admitted to the Union since the adoption of the Constitution?

39. What is the excess of the area of the Middle States over that of the New England States? (See Art. 34, Ex. 22.)

40. The distance from the earth to the sun is about 95,000,000 miles; the distance to the moon is about 240,000. How much farther to the sun than to the moon?

2*

38. Examples in Addition and Subtraction.

1. A bought 113 acres of land of B, 254 acres of C, 74 acres of D and 396 acres of E, and afterwards sold 75 acres to F, 206 acres to G, gave 150 acres to his oldest son, 100 acres to his second son, and retained the remainder. What was the largest number of acres owned by A? how many acres did he sell? how many did he give away? and how many did he keep?

2. How many are $3694 + 78769 - 354 + 876 + 4327 - 869 - 473 - 63 + 56 - 6 + 87 - 14$?

3. How much less was the population of England in 1851 than that of the United States in 1850? (See Art. 34, Ex. 19 and 20.)

4. The estimated expenses of the United States Coast Survey for the fiscal year 1852–3, were, for the coast of Me., N. H., Mass. and R. I. \$36000; N. J., Pa. and Del. \$7000; Md. and Va. \$33000; N. C. \$25000; S. C. and Ga. \$23000; Ala., Miss and La. \$25000; and for Texas \$21000. Suppose the expenses for surveying this extensive coast to be the same for the four succeeding years, what will be the cost for five years? how much less than the cost of the American army for the five years 1812–16? (Art. 34, Ex. 31.)

5. Methuselah lived 969 years; how much longer is that than from the settlement of Boston in 1630 to the present time?

6. Adam lived 930 years; Seth, 912; Enos, 905; Cainan, 910; Jared, 962; Methuselah, 969; Noah, 950. Washington died at the age of 67 years; J. Adams, at 91; Jefferson, 84; Madison, 85; Monroe, 72; J. Q. Adams, 81; Jackson, 78. What is the aggregate age of the seven antediluvians mentioned? the aggregate age of the first seven Presidents of the United States? what the difference of the aggregate age of the seven antediluvians and that of the seven Presidents?

7. The population of these five African cities in 1850 was about as follows, viz. Cairo 240,000, Alexandria 35,000, Tripoli 25,000, Tunis 150,000 and Algiers (in 1840) 40,000; that of these five Asiatic cities, as follows, viz. Pekin 1,750,000, Nankin

(in 1851) 800,000, Canton 800,000, Calcutta 1,580,000 and Bombay 235,000. What was the total population of the African cities, and how much greater was that of the Asiatic cities?

8. What was the aggregate population of the following sixteen European cities in 1852, and what the difference between this aggregate and that of the sixteen named cities in the United States in 1850? — viz.

Europe.		United States.	
London,	2,363,141	New York,	515,547
Paris, (in 1846,)	1,053,897	Philadelphia,	340,045
Constantinople,	786,990	Baltimore,	169,054
St. Petersburg,	478,437	Boston,	136,881
Vienna,	477,846	New Orleans,	116,375
Berlin,	441,931	Cincinnati,	115,436
Naples,	416,475	Brooklyn,	96,838
Liverpool,	384,265	St. Louis,	77,860
Glasgow,	367,800	Albany,	50,763
Moscow,	350,000	Pittsburgh,	46,601
Manchester,	296,000	Louisville,	43,194
Madrid,	260,000	Charleston,	42,985
Dublin,	254,850	Buffalo,	42,261
Lyons,	249,325	Providence,	41,513
Lisbon,	241,500	Washington,	40,001
Amsterdam,	222,800	Newark,	38,894

9. The populations of the leading European countries at the middle of the nineteenth century were as follows, viz. Great Britain and Ireland 27,332,145, France 35,783,170, Russia 62,088,000, Austria 36,514,397, Prussia 16,331,187, Spain 12,232,194, Turkey 12,000,000; what was the total population of those countries, and what the difference between the population of each of them and that of the United States? (See Art. 34, Ex. 10.)

§ 5. MULTIPLICATION.

39. MULTIPLICATION * is a short method of *adding equal numbers;* i. e. *multiplication* is a short method of *finding the sum of the repetitions* of a number which is repeated as many times as there are *units* in another number.

The *number to be repeated* is the *multiplicand;* the *number showing how many times* the multiplicand is to be repeated, is the *multiplier;* the *sum* or *result* of the multiplication is the *product.* The *multiplicand* and *multiplier* are called *factors.*†

Ex. 1. In one bushel are 32 quarts; how many quarts in 6 bushels?

BY ADDITION.	BY MULTIPLICATION.
3 2	3 2
3 2	6
3 2	Product, 1 9 2
3 2	
3 2	
3 2	
Sum, 1 9 2	

In six bushels there are, evidently, 6 times as many quarts as in 1 bushel, and the number of quarts in 6 bushels may be obtained by *adding*, as in the margin; or, more briefly, by *multiplying;* thus, 6 times 2 units are 12 units = 1 ten and 2 units; write the 2 units in units' place, and then say 6 times 3 tens are 18 tens, which, increased by the 1 ten previously obtained, make 19 tens = 1 hundred and 9 tens, and these, written in the place of hundreds and tens respectively, give the true product.

Ex. 2. How many quarts in 46 bushels?

OPERATION.	
Multiplicand,	3 2
Multiplier,	4 6
	1 9 2
	1 2 8
Product,	1 4 7 2

First multiply by 6 as though 6 were the only figure in the multiplier; then multiply by 4 and set the first figure of the product in the place of *tens;* for multiplying by the 4 *tens* is the same as multiplying by 40, and 40 times 2 units are 80 *units* = 8 *tens;* i. e. the product of *units* by *tens* is *tens.* Having multiplied by each figure in the multiplier, the *sum* of the *partial*

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* *Multiplication,* from the Latin *multiplico,* (*multus, many,* and *plico, to fold,*) *to fold many times.*

† *Factor,* from the Latin *facio, to make, to produce.*

products will be the *true product.* Similar reasoning applies however many figures there may be in the multiplier.

40. From these examples we derive the following

RULE. — 1. *Set the multiplier under the multiplicand and draw a line beneath.*

2. *Beginning at the right hand of the multiplicand, multiply the multiplicand by each figure in the multiplier, setting the first figure of each partial product directly under the figure of the multiplier by which that product is obtained.*

3. *The* SUM *of these partial products will be the true product.*

41. PROOF. — *It is not material which factor is taken for multiplier, ∴ make each, in turn, the multiplier, and* THE TWO PRODUCTS WILL BE ALIKE.

Ex. 3. Multiply 3642 by 1758.

	OPERATION.	PROOF.
Multiplicand,	3642	1758
Multiplier,	1758	3642
	29136	3516
	18210	7032
	25494	10548
	3642	5274
Product,	6402636	6402636

	4.	5.
Multiplicand,	47869	78694327
Multiplier,	85642	2789

6.	7.	8.
84279548	896542	467893254
42639742	369	876954732695

9. Multiply 6742 by 967. Ans. 6519514.

10. Multiply 528764239 by 349.

11. Multiply 279864327 by 2789.

12. $634278 \times 23 =$ how many? Ans. 14588394
13. $98742968 \times 2791 = ?$ Ans. 275591623688
14. $4698273 \times 1913 = ?$
15. $5678 \times 38769542 = ?$
16. $123456789 \times 987654321 = ?$
17. $9999999999 \times 9999999999 = ?$
18. $3333333 \times 3333333 = ?$
19 $55555555 \times 66666666 = ?$
20 $913465728 \times 271936485 = ?$
21. $421693578 \times 875329416 = ?$
22. $167757216 \times 466578324 = ?$
23. $44556677889 \times 98887766554 = ?$
24. $841784729676 \times 576529484441 = ?$
25. $248163264128 \times 256289324361 = ?$

42. The rule already given is applicable in all examples that can arise in multiplication, but there are various devices for shortening the process in particular cases.

43. The product of two or more whole numbers greater than 1 is called a *composite number;* the *factors* are called the *component parts;* thus, 12 is a composite number, of which 2 and 6, 3 and 4 or 2, 2 and 3 are component parts or factors.

44. When the multiplier is a composite number, multiply the multiplicand by one of the factors of the multiplier and that product by another factor, and so on until all the factors have been taken; the *last* product will be the *true* product.

Ex. 26. Multiply 37 by 35.

OPERATION.

$35 = 5 \times 7.$

Multiplicand,	37
1st Factor of Multiplier,	5
	185
2d Factor of Multiplier,	7
Product,	1295

It is evident that 7 times 5 times a number are 35 times that number

Ex. 27. Multiply 293 by 125.

Multiplicand,	2 9 3
1st Factor,	5
	1 4 6 5
2d Factor,	5
	7 3 2 5
3d Factor,	5
Product,	3 6 6 2 5

The first method is frequently preferable to this, even when the multiplier is composite.

28. Multiply 743 by 42, i. e. by 7 and 6. Ans. 31206.
29. Multiply 3467 by 56.
30. $839 \times 54 =$ how many?
31. $7869 \times 72 = ?$
32. $469876 \times 81 = ?$
33. $478969 \times 1728 = ?$
34. $5387469 \times 96 = ?$
35. $987462 \times 49 = ?$

45. A number is multiplied by 10 by annexing 0 to it, for, by so doing, each figure of the multiplicand is removed one place towards the left, and thus its value is made tenfold (Art. 16). For a like reason a number is multiplied by 100, 1000, etc., by annexing as many ciphers to the multiplicand as there are ciphers in the multiplier.

Ex. 36. Multiply 74 by 10. Ans. 740.
37. Multiply 869 by 10000. Ans. 8690000.
38. Multiply 4698 by 1000.
39. $76984 \times 100000 = ?$ Ans. 7698400000.
40. $59874 \times 1000000000 = ?$

46. To multiply by 20, 50, 500, 25000, or any similar number, *multiply by the significant figures and to the product annex as many ciphers as there are ciphers at the right of the significant figures of the multiplier.*

Ex. 41. Multiply 756 by 30.

OPERATION.

```
  7 5 6
    3 0
 ------
2 2 6 8 0
```

This is upon the principle of Art. 44. The component parts of 30 are 3 and 10. Having multiplied by 3, the product is multiplied by 10 by annexing 0 (Art. 45).

42. Multiply 743 by 3500.

```
          7 4 3
              7
        -------
        5 2 0 1
            5 0 0
        ---------
Product, 2 6 0 0 5 0 0
```

The component parts of 3500 are 7, 5 and 100.

43. Multiply 84693 by 480000. Ans. 40652640000.

44. $8769432 \times 7200000 = ?$

45. $94684235 \times 49000000 = ?$

47. Not only may the *multiplier* be factored, but, upon the same principle, the component parts of the *multiplicand* may be taken separately. This is convenient when there are ciphers at the right of the multiplicand.

Ex. 46. Multiply 8000 by 900.

```
          8 0 0 0
          9 0 0
         -------------
Product, 7 2 0 0 0 0 0
```

The factors of 8000 are 8 and 1000, and those of 900 are 9 and 100. We multiply the significant figures of the two numbers together and to the product annex as many ciphers as there are ciphers at the right of the multiplicand and multiplier counted together.

47 Multiply 730000 by 2900.

OPERATION.

```
            7 3 0 0 0 0
            2 9 0 0
          -------------
          6 5 7
        1 4 6
        -----------------
Product, 2 1 1 7 0 0 0 0 0 0
```

48. Multiply 8,400 by 2,700,000. Ans. 22,680,000,000.

49. 7,693,000 × 569,000 = ?

50. 8,769,432,000 × 48,700 = ?

48. Ciphers between the significant figures of the multiplier may be neglected, taking care to set the first figure of each partial product directly under the figure of the multiplier which gives that product.

Ex. 51. Multiply 5,723 by 2,004.

OPERATION.

```
                 5,7 2 3
                 2,0 0 4
                 -------
               2 2 8 9 2
           1 1 4 4 6
           -------------
Product,   1 1,4 6 8,8 9 2
```

This is only carrying out the principle (in addition) of setting units under units, tens under tens, etc. The 2 of the multiplier is 2000, and 2000 times 3 are 6000, ∴ the 6 of the partial product should be written in the thousands' place, i. e. directly under the 2 of the multiplier.

52. Multiply 3724 by 4008. Ans. 14925792.

53. 698427 × 420006 = ?

54. 58067082 × 3923007 = ?

55. 7800076900 × 200804000 = ?

56. What cost 11 pounds of beef at 14 cents per pound?

EXPLANATION. — 11 pounds will cost 11 times as many cents as 1 pound, ∴, since 1 pound costs 14 cents, 11 pounds will cost 11 times 14 cents = 154 cents = $1.54, Ans.

57. What cost 98 tons of hay at $15 per ton? Ans. $1470.

58. In one hogshead of wine are 63 gallons; how many gallons in 75 hogsheads?

59. In a certain house are 75 rooms, in each room 4 windows, in each window 12 panes of glass and in each pane 120 square inches; how many square inches of glass in the house?

60. The earth, in its annual revolution, moves 19 miles per second; how far will it move in 1 week — there being 7 days

in 1 week, 24 hours in 1 day, 60 minutes in 1 hour and 60 seconds in 1 minute? Ans. 11491200 miles.

61. What is the value of 379 acres of land at $153 per acre?

62. Two men start from the same place and travel in the same direction; one travels 56 miles and the other 75 miles per day; how far apart will they be at the end of 43 days?

63. Had the men named in Ex. 62 traveled in opposite directions, how far apart would they have been in 56 days?

64. How many yards of cloth in 43 bales, each bale containing 53 pieces and each piece 34 yards?

65. What is the value of the cloth mentioned in Ex. 64, at $3 per yard? Ans. $232458.

66. 53 men can do a piece of work in 72 days; in how many days can 1 man do 25 times as much work?

67. What is the value of 36 cords of wood at $6 per cord, 752 barrels of flour at $12 per barrel, 1000 bushels of potatoes at $1 per bushel, 10 oxen at $73 per ox and 5 horses at $125 per horse?

68. If 1 man earn $11 in 1 week, how many dollars will 17 men earn in 52 weeks?

69. The submarine telegraph cable, now preparing (May, 1857) to connect Europe and America, is composed of 7 copper wires, imbedded in gutta percha, surrounded by 18 bundles of iron wire, each bundle composed of 7 wires; how many wires are there in the cable? Ans. 133.

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§ 6. DIVISION.

49. DIVISION is the process of finding how many times one number is contained in another.

The *number to be divided* is called the *dividend*, the *number by which to divide*, the *divisor;* the *result*, the *quotient; if any thing is left after dividing*, the *remainder.*

50. *The remainder is always of the same kind as the dividend;* e. g. if the *dividend* is *miles* the *remainder* is *miles;* if the *dividend* is *dollars* the *remainder* is *dollars;* etc.

Ex. 1. Divide 1384 by 4.

OPERATION.

```
4)1384(346
  12
  --
   18
   16
   --
    24
    24
    --
     0
```

Having written the divisor and dividend as in the margin, we first inquire how many times 4 is contained in 13, (the fewest figures at the left of the dividend that will contain the divisor,) and find the quotient to be 3, which we set at the right of the dividend. We then multiply the divisor by the quotient, 3, and set the product, 12, under the 13 of the dividend and subtract it therefrom. To the remainder, 1, we annex 8, the next figure of the dividend, and then inquire how many times the divisor is contained in 18, the second partial dividend; the result, 4, we set as the second figure of the quotient and then multiply, subtract, annex, etc. as before, until all the figures of the dividend have been taken.

51. Since the 13 of the dividend is *hundreds*, the 3 of the quotient is *also hundreds;* since the 18 is *tens*, the 4 is *also tens;* and, *universally, any quotient figure is of the same order as the right-hand figure of the dividend taken to obtain that quotient figure.*

52. The operation may be much shortened by *carrying the process in the mind, instead of writing it;* thus, having written the divisor and dividend as before, say 4 in 13, 3 times and 1 remainder; set the quotient, 3, under the 3 of the dividend, and then, *imagining* the remainder, 1, *set before* the 8, say 4 in 18, 4 times and 2 remainder; set down the 4 as the second figure of the quotient and imagine the 2 set before the next figure, and so proceed.

```
Divisor, 4)1384 Dividend.
Quotient,  346
```

53. The operation in Art. 50 is called *Long Division;* that in Art. 52 is *Short Division,* which may be performed by the following

RULE. — *Divide the left-hand figure or figures of the dividend, (the fewest figures in the dividend that will contain the divisor,) and set the quotient under the right-hand figure considered in the dividend; if anything remains, prefix it* MENTALLY *to the next figure in the dividend and divide the number thus formed as before, and so proceed till all the figures of the dividend have been employed.*

Ex. 2. Divide 2776 by 8.

OPERATION.

$$\begin{array}{rl} \text{Divisor, } 8\,) & \underline{2\;7\;7\;6} \text{ Dividend.} \\ \text{Quotient, } & \;\;3\;4\;7 \end{array}$$

Ex. 3. Divide 2781 by 8.

OPERATION.

$$\begin{array}{rl} \text{Divisor, } 8\,) & \underline{2\;7\;8\;1} \text{ Dividend.} \\ \text{Quotient, } & \;\;3\;4\;7 \ldots 5 \text{ Remainder.} \end{array}$$

4. Divide 3283 by 7. Ans. Quo. 469.
5. Divide 4050 by 7. Ans. Quo. 578, Rem. 4.
6. Divide 476589 by 9.
7. $987654 \div 12 = ?$ Ans. Quo. 82304, Rem. 6.
8. $59684 \div 4 = ?$
9. $87695425 \div 5 = ?$
10. $7869468923 \div 11 = ?$

54. When the divisor is large, it is not convenient to carry the process in the mind, but the work may be performed, in *long division,* by the following

RULE. — 1. *Write the divisor and dividend as in short division and draw a curved line at the right of the dividend.*

2. *Divide the smallest number of figures in the left of the*

dividend that will contain the divisor and set the result as the first figure of the quotient at the right of the dividend.

3. *Multiply the divisor by the quotient figure and set the product under that part of the dividend taken.*

4. *Subtract the product from the figures over it and to the remainder annex the next figure of the dividend for a new partial dividend.*

5. *Divide and proceed as before until the whole dividend has been divided.*

NOTE 1. — It will be seen that the process of dividing consists of four distinct steps, viz., first, to seek a quotient figure; second, multiply; third, subtract; and, fourth, form a new partial dividend by annexing the next figure of the dividend to the remainder.

NOTE 2. — If any partial dividend will not contain the divisor, 0 must be placed in the quotient and another figure annexed to the dividend.

NOTE 3. — If the product of the quotient figure multiplied by the divisor is greater than the partial dividend, the quotient figure is too large and must be diminished.

NOTE 4. — If the remainder equals or exceeds the divisor, the quotient is too small and must be increased.

55. Division is the reverse of multiplication. In multiplication the two factors are given, and the product is required; in division the product and one factor are given, and the other factor is required. The dividend is the product, and the divisor and quotient are the factors. Hence the

PROOF. — *Multiply the divisor by the quotient and to the product add the remainder; the* SUM *should be the dividend.*

Ex. 11. Divide 3413 by 63.

OPERATION.

```
63)3413(54
   315
   ---
    263
    252
    ---
     11
```

PROOF.

```
  63 Divisor.
  54 Quotient.
 ---
 252
315
  11 Remainder.
----
3413 Dividend.
```

12. Divide 78946287 by 3498.

Ans. Quo. 22568, Rem. 3423

13. Divide 1764842 by 347.

14. $896842 \div 547 = ?$ Ans. Quo. 1639, Rem. 309.

15. $569432 \div 45 = ?$

16. $98647324 \div 4893 = ?$

17. $698742346525 \div 6995 = ?$

56. When the divisor is a composite number, the work may often be abridged by dividing first by one factor of the divisor, and the quotient thus obtained by another, and so on till all the factors have been used; the last quotient is the quotient sought.

Ex. 18. Divide 1855 by 35.

OPERATION.

$35 = 7 \times 5.$

1st Factor,	7)	1 8 5 5	Dividend.
2d Factor,	5)	2 6 5	1st Quotient.
		5 3	True Quotient.

This operation is evidently correct, for one fifth of one seventh of any number is the same as one thirty-fifth of that number.

19. Divide 1551 by 33. Ans. 47.

20. Divide 31794 by 42.

21. Divide 47936 by 56.

22. Divide 24840 by 72.

23. Divide 7665 by 105.

OPERATION.

$105 = 3 \times 5 \times 7.$

3)	7 6 6 5
5)	2 5 5 5
7)	5 1 1
Quotient,	73

(a) Sometimes a composite number is made up of different sets of factors, as in Ex. 24. When this is the case, it is immaterial which set is taken.

24. Divide 22320 by 240.

$240 = 8 \times 6 \times 5 = 4 \times 12 \times 5 = 4 \times 6 \times 10 =$ etc.

FIRST OPERATION.		SECOND OPERATION.
8) 2 2 3 2 0		4) 2 2 3 2 0
6) 2 7 9 0		1 2) 5 5 8 0
5) 4 6 5		5) 4 6 5
Quotient, 9 3	=	9 3

25. Divide 187236 by 252. Ans. 743.

26. Divide 1255872 by 192.

57. Should the learner find a difficulty in determining the remainder, he has but to remember that it is *always* of the same kind as the dividend (Art. 50).

27. Divide 86 by 21.

OPERATION.

7) 8 6
3) 1 2 . . . 2 Rem.
Quotient, 4

In this example, as 86 is the true dividend, 2 is the true remainder.

28. Divide 92 by 28.

OPERATION.

4) 9 2
7) 2 3
Quotient, 3 . . . 2 Rem.

In this example, as 23 is only one fourth of the true dividend, so the remainder, 2, is only one fourth of the true remainder, $\therefore$ $2 \times 4 = 8$, true remainder.

29. Divide 527 by 42.

OPERATION.

6) 5 2 7
7) 8 7 . . . 5 Rem.
Quotient, 1 2 . . . 3 Rem.

By the explanations of the last two examples, we see that 5 is one part of the true remainder, and that 3, the second remainder, multiplied by 6, the first divisor, is the other part; i. e. $5 + 3 \times 6 = 23 =$ true remainder. The same species of reasoning applies when there are more than two divisors. Hence,

To obtain the true remainder when division is performed by using the component parts of the divisor,

RULE 1. — *Multiply each remainder except that left by the first division, by the continued product of the divisors preceding that which gave the remainders severally, and the sum of the products together with the remainder left by the first division will be the true remainder.*

30. Divide 15956 by 280.

OPERATION.	TRUE REMAINDER.
$280 = 7 \times 5 \times 8.$	$3 =$ 1st Rem.
7) 1 5 9 5 6	$4 \times 7 = 28 =$ 1st Prod.
5) 2 2 7 9 ... 3 Rem.	$7 \times 5 \times 7 = 245 =$ 2d Prod.
8) 4 5 5 ... 4 Rem.	$276 =$ True Rem.
Quo. 5 6 ... 7 Rem.	

RULE 2. — *Multiply the last remainder by the divisor preceding that which gave the last remainder and to the product add the preceding remainder; multiply this sum by the preceding divisor and add the preceding remainder, and so proceed until the first remainder is added; the sum so obtained will be the true remainder.*

Should any remainder be 0, then 0 is to be added.

APPLICATION OF THIS RULE TO EX. 30. $7 \times 5 + 4 = 39$; $39 \times 7 + 3 = 276$, true remainder, as by Rule 1.

31. Divide 5273 by 42.
$42 = 2 \times 3 \times 7.$ Ans. 125 and 23 Rem.

32. Divide 46987 by 504, using the factors of the divisor.
Ans. 93 and 115 Rem.

33. Divide 78925 by 105.

34. $437298 \div 154 = ?$ Ans. 2839 and 92 Rem.

35. $216349 \div 315 = ?$

36. $2411 \div 385 = ?$

37. $36067 \div 4199 = ?$

58. To divide by 10, cut off, by a point, one figure from the right of the dividend; the figures at the left of the point are the quotient, and that at the right is the remainder.

The reason is obvious; by taking away the right-hand figure, each of the other figures is brought one place nearer to units, and its value is only one tenth as great as before (Art. 16), ∴ the whole is divided by 10.

38. Divide 756 by 10. Ans. 75.6, i. e. 75 Quo. and 6 Rem.
39. Divide 402763 by 10.

(a) For like reasons we cut off *two* figures to divide by 100, *three* to divide by 1000, and, generally,* we cut off as many figures from the right of the dividend as there are ciphers in the divisor.

40. Divide 76943 by 100. Ans. 769 and 43 Rem.
41. Divide 98765423 by 100000.
Ans. 987 and 65423 Rem.
42. Divide 3078654321 by 100000000.

(b) To divide by 20, 50, 700, 56000, or any similar number, cut off as many figures from the right of the dividend as there are ciphers at the right of the significant figures of the divisor, and then divide the remaining figures of the dividend by the significant figures of the divisor. This is on the principle of dividing by the component parts of the divisor, ∴ the true remainder will be found by the rules in Art. 57.

43. Divide 74689 by 8000. Ans. 9 and 2689 Rem.

OPERATION.

$$8\,)\,74.689$$

Quotient, 9 . . . 2 Rem.

We divide by 1000 by cutting off 689, which gives 74 for a quotient and 689 for a remainder; then divide 74 by 8 and obtain the quotient, 9, and remainder, 2. This remainder, 2, is 2000, which, increased by 689, gives 2689 for the true remainder (Art. 57, Rule 2).

* *Generally*, in mathematics, means *universally*.

44. Divide 67475 by 2400.
45. Divide 74689 by 4200. Ans. 17 and 3289 Rem.
46. Divide 276987 by 3300.
47. $769842 \div 45000 = ?$ Ans. 17 and 4842 Rem.
48. $9999999 \div 33300 = ?$
49. $80407080 \div 40000 = ?$
50. $987654321 \div 90900 = ?$
51. $9876543210 \div 909000 = ?$
52. $123456789 \div 90900 = ?$

53. A certain product is 1728, and one of the factors is 12; what is the other factor? Ans. 144.

54. How many times is 157 contained in 74732?

55. By what must 316128 be divided to give 356 for a quotient?

56. By what must 87 be multiplied to produce 83868?

57. If 1357901 be a dividend and 87 the divisor, what is the quotient? remainder?

58. Dividend = 6789468; quotient = 1234; divisor = ?

59. A dividend is 6481, the quotient is 72 and the remainder is 73; what is the divisor?

60. Dividend = 98765; divisor = 17; remainder = ?

61. The product of three numbers is 16777216, and the product of two of them is 131072; what is the other number?

62. $248832 = 144 \times ?$

59. The value of a quotient depends upon the *relative* values of the divisor and dividend and not upon their *absolute* values, as will be seen by the following propositions.

(a) If the divisor remains unaltered, multiplying the dividend by any number is, in effect, multiplying the quotient by the same number; thus,

$$\begin{array}{rcrcr} 15 & \div & 3 & = & 5 \\ \underline{4} & & & & \underline{4} \\ 60 & \div & 3 & = & 20; \end{array}$$

i. e. multiplying the dividend by 4 multiplies the quotient by 4.

(b) Dividing the dividend by any number is dividing the quotient by the same number; thus,

$$\begin{array}{rl} & 24 \div 2 = 12 \\ 3\,) & \underline{24} \\ & \;\;8 \div 2 = \;\;4 = 12 \div 3; \end{array}$$

i.e. dividing the dividend by 3 divides the quotient by 3.

(c) Multiplying the divisor divides the quotient; thus,

$$\begin{array}{l} 30 \div \;2 = 15 \\ \quad\;\;\;\underline{\;3\;} \\ 30 \div \;6 = \;\;5 = 15 \div 3; \end{array}$$

i. e. multiplying the divisor by 3 divides the quotient by 3.

(d) Dividing the divisor multiplies the quotient; thus,

$$\begin{array}{l} 40 \div 10 = \;\;4 \\ \quad\;\; 5\,)\,\underline{10} \\ 40 \div \;\;\;2 = 20 = 4 \times 5; \end{array}$$

i. e. dividing the divisor by 5 multiplies the quotient by 5.

(e) Corollary* to (a) and (b). — The greater the dividend the greater the quotient, and the reverse.

(f) Corollary to (c) and (d). — The greater the divisor the less the quotient, and the reverse.

60. If a number be multiplied by any number, and the product be divided by the multiplier, the quotient will be the multiplicand; thus,

$8 \times 7 = 56$, and $56 \div 7 = 8$, the multiplicand.

Corollary. — Since multiplying the dividend multiplies the quotient (59, a), and multiplying the divisor divides the quotient (59, c), $\therefore$ *multiplying both dividend and divisor by the same number does not affect the quotient;* thus,

$$\begin{array}{l} 12 \div 3 = 4 \\ \underline{\;\;2\;} \quad \underline{2} \\ 24 \div 6 = 4, \text{ Quotient unchanged.} \end{array}$$

* A *corollary* is an inference from preceding reasoning.

61. If a number be divided by any number, and the quotient be multiplied by the divisor, the product will be the dividend; thus,

$$15 \div 3 = 5, \text{ and } 5 \times 3 = 15, \text{ the dividend.}$$

Corollary. — Since dividing the dividend divides the quotient (59, b), and dividing the divisor multiplies the quotient (59, d), ∴ *dividing both dividend and divisor by the same number does not affect the quotient;* thus,

$$\begin{array}{rcrl} 20 & \div & 10 & = 2 \\ 5)\underline{20} & & 5)\underline{10} & \\ 4 & \div & 2 & = 2, \text{ Quotient unchanged.} \end{array}$$

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§ 7. COMPOUND NUMBERS.

62. Numbers are either *Simple* or *Compound.*

63. *A Simple Number* consists of but *one kind* or *denomination;* thus, 3, 15, 8 books, 4 men, 7 apples, etc. are simple numbers.

Note. — All operations in the preceding pages are upon simple numbers.

64. *A Compound Number* is composed of *two or more denominations;* thus, 4 days and 7 hours; 3 bushels, 2 pecks and 5 quarts; etc., are compound numbers.

Note. — The several parts of a compound number, though of *different denominations,* are yet of the *same general nature;* thus, 2 weeks, 3 days and 6 hours are SIMILAR *quantities and constitute a compound number;* but 2 weeks, 3 miles and 6 quarts are UNLIKE IN THEIR NATURE *and do* NOT *constitute a compound number.*

65. The first division of each of the following tables should now be THOROUGHLY *committed to memory.*

66. FEDERAL MONEY.

10 Mills (m.)	make	1 Cent,	marked	c.
10 Cents	"	1 Dime,	"	d.
10 Dimes	"	1 Dollar,	"	$
10 Dollars	"	1 Eagle,	"	e.

Eagle		Dollars.		Dimes.		Cents.		Mills
						1	=	10
				1	=	10	=	100
		1	=	10	=	100	=	1000
1	=	10	=	100	=	1000	=	10000

NOTE 1. — Federal Money is the National Currency of the United States.

NOTE 2. — The terms, eagle and dime, are seldom used in computation; eagles and dollars being read collectively and called dollars, and dimes and cents being called cents; thus, 3 eagles and 5 dollars are called $35, and 4 dimes and 3 cents are called 43 cents.

67. ENGLISH MONEY.

4 Farthings (qr.)	make	1 Penny,	d.
12 Pence	"	1 Shilling,	s.
20 Shillings	"	1 Pound,	£.

£		s.		d.		qr.
				1	=	4
		1	=	12	=	48
1	=	20	=	240	=	960

NOTE. — English Money is the National Currency of Great Britain.

68. TROY WEIGHT.

24 Grains (gr.)	make	1 Pennyweight,	dwt.
20 Pennyweights	"	1 Ounce,	oz.
12 Ounces	"	1 Pound,	lb.

lb.		oz.		dwt.		gr.
				1	=	24
		1	=	20	=	480
1	=	12	=	240	=	5760

NOTE. — Troy weight is used in weighing gold, silver and precious stones.

4

69. APOTHECARIES' WEIGHT.

20 Grains (gr.)	make	1 Scruple,	℈
3 Scruples	"	1 Dram,	ʒ
8 Drams	"	1 Ounce,	℥
12 Ounces	"	1 Pound,	lb.

						sc.		gr.
				dr.		1	=	20
		oz.		1	=	3	=	60
lb.		1	=	8	=	24	=	480
1	=	12	=	96	=	288	=	5760

NOTE 1.—Medicines are *mixed* or *compounded* by this weight, but are usually bought and sold by Avoirdupois weight.

NOTE 2.—The pound, ounce and grain in Apothecaries' and Troy weight are equal, but the ounce is differently subdivided.

70. AVOIRDUPOIS WEIGHT.

16 Drams (dr.)	make	1 Ounce,	oz.
16 Ounces	"	1 Pound,	lb.
25 Pounds	"	1 Quarter,	qr.
4 Quarters	"	1 Hundred Weight,	cwt.
20 Hundred Weight	"	1 Ton,	t.

								oz.		dr.
						lb.		1	=	16
				qr.		1	=	16	=	256
		cwt.		1	=	25	=	400	=	6400
t.		1	=	4	=	100	=	1600	=	25600
1	=	20	=	80	=	2000	=	32000	=	512000

NOTE 1.—The coarser articles of merchandise, such as hay, cotton, tea, sugar, copper, iron, etc., are weighed by Avoirdupois weight.

NOTE 2.—It was the custom, formerly, to consider 28 lbs. a quarter, 112 lbs. a cwt., and 2240 lbs. a ton; but now the *usual* practice is in accordance with the table.

These different tons are distinguished as the *long* or *gross* ton = 2240 lbs., and the *short* or *net* ton = 2000 lbs.

NOTE 3.—A pound in Avoirdupois weight is equal to 7000 grains in Apothecaries' or Troy weight.

71. CLOTH MEASURE.

$2\frac{1}{4}$* Inches (in.)	make	1 Nail,	na.
4 Nails	"	1 Quarter,	qr.
4 Quarters	"	1 Yard,	yd.

				na.		in.
		qr.		1	=	$2\frac{1}{4}$
yd.		1	=	4	=	9
1	=	4	=	16	=	36

NOTE. — Cloths of every description, ribbons, braids, etc. are measured by this measure.

72. LONG MEASURE.

3 Barleycorns (b. c.)	make	1 Inch,	in.
12 Inches	"	1 Foot,	ft
3 Feet	"	1 Yard,	yd.
$5\frac{1}{2}$ Yards or $16\frac{1}{2}$ Feet,	"	1 Rod,	rd.
40 Rods	"	1 Furlong,	fur.
8 Furlongs	"	1 Mile,	m.
3 Miles	"	1 League,	l.
$69\frac{1}{2}$ Statute Miles, nearly,	"	1 Degree on Circ. of the Earth,	1°
360 Degrees	"	1 Circumference,	circ.

										in.		b. c.
								ft.		1	=	3
						yd.		1	=	12	=	36
				rd.		1	=	3	=	36	=	108
		fur.		1	=	$5\frac{1}{2}$	=	$16\frac{1}{2}$	=	198	=	594
m.		1	=	40	=	220	=	660	=	7920	=	23760
1	=	8	=	320	=	1760	=	5280	=	63360	=	190080

NOTE. — This measure is used in measuring distances, i. e. where length is required without regard to breadth or thickness.

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* Expressions like $\frac{1}{4}$, $\frac{2}{3}$, etc. are called *fractions*. $\frac{1}{4}$ = *one fourth*; $\frac{2}{3}$ = *two thirds*; $2\frac{1}{4}$ = *two and one-fourth*. The principles of fractions will be discussed in §§ 10 and 11.

73. CHAIN MEASURE.

$7\frac{92}{100}$	Inches (in.)	make	1 Link,	li
25	Links	"	1 Rod, Perch or Pole,	rd.
4	Rods	"	1 Chain,	ch.
10	Chains	"	1 Furlong,	fur
8	Furlongs	"	1 Mile,	m.

m.		fur.		ch.		rd.		li.		in.
								1	=	$7\frac{92}{100}$
						1	=	25	=	198
				1	=	4	=	100	=	792
		1	=	10	=	40	=	1000	=	7920
1	=	8	=	80	=	320	=	8000	=	63360

NOTE. — This measure is used by engineers in measuring roads, canals etc.; also by surveyors in measuring the boundaries of fields.

74. SQUARE MEASURE.

144	Square Inches (sq. in.)	make	1 Square Foot,	sq. ft
9	Square Feet	"	1 Square Yard,	sq. yd.
$30\frac{1}{4}$	Square Yards or	"	1 Square Rod,	sq. rd.
$272\frac{1}{4}$	Square Feet			
40	Square Rods	"	1 Rood,	r.
4	Roods	"	1 Acre,	a.
640	Acres	"	1 Square Mile,	sq. m.

(a) Also, in Chain Measure,

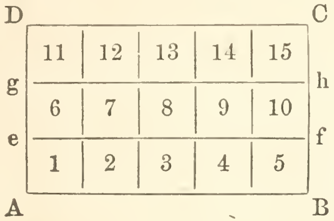
10000	Square Links or	make	1 Square Chain,	sq. ch.
16	Square Rods			
10	Square Chains	"	1 Acre,	a.

sq m.	a.	r.	sq. rd.	sq. yd.	sq. ft.	sq. in.
					1=	144
				1=	9=	1296
			1=	$30\frac{1}{4}$=	$272\frac{1}{4}$=	39204
		1=	40=	1210=	10890=	1568160
	1=	4=	160=	4840=	43560=	6272640
1=	640=	2560=	102400=	3097600=	27878400=	4014489600

NOTE 1. — This measure is used in measuring surfaces.

NOTE 2. — In measuring land, surveyors use a 4-rod chain composed of 100 links. Sometimes the half-chain of 50 links is used.

75. The manner of determining the area of a surface like the marginal figure, may be understood from the following explanation. Let A B represent (on a reduced scale) a line 5 inches in length: then, evidently, if we pass from A to e, a distance of 1 inch, and draw the line e f, the figure A B f e will contain 5 square inches, i. e. 5 × 1 square inches.



In like manner A B h g will contain 10, or 5 × 2 square inches; and A B C D will contain 15, or 5 × 3 square inches; i. e. we multiply the numbers expressing the length and breadth together, and the product will be the number of square inches in the surface.

(a) Likewise, the *area divided* by the *length* will give the *breadth*, and the *area divided* by the *breadth* will give the *length;* thus, $15 \div 5 = 3$, and $15 \div 3 = 5$.

76. SOLID OR CUBIC MEASURE.

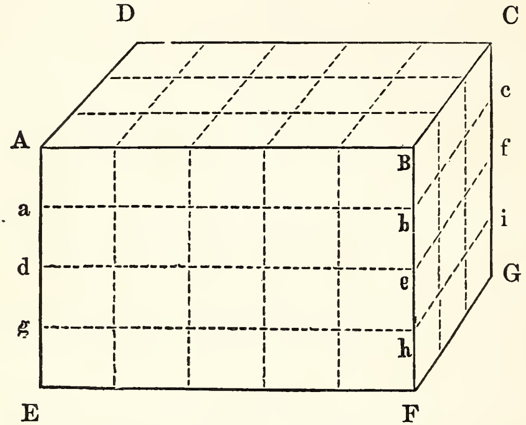
1728 Cubic Inches (c. in.)	make	1 Cubic Foot,	cu. ft.
27 Cubic Feet	"	1 Cubic Yard,	c. yd.
40 Cubic Feet	"	1 Ton of Timber,	t.
16 Cubic Feet	"	1 Cord Foot,	c. ft.
8 Cord Feet or 128 Cubic Feet	"	1 Cord,	c.

c. yd.		cu. ft.		c. in.
		1	=	1728
1	=	27	=	46656

NOTE. — This table is used in measuring things which have length, breadth and thickness.

77. To determine the contents of a body in the form of the following figure, first find the area of the upper surface,

A B C D, as in Art. 75; then going from A, B and C downward 1 inch to a, b and c, and passing a plane through a, b and c, we shall cut off 15 solid inches, i. e. 5 × 3 × 1 solid inches



So if a plane be passed through d, e and f, it will cut off 30, or 5 × 3 × 2 inches, etc.; i. e. the continued product of the numbers expressing the length, breadth and depth, will give the solid contents.

(a) So, also, the solid contents divided by the area of the top face will give the depth.

What are the solid contents of the above figure?

78. LIQUID MEASURE.

4 Gills (gi.)	make	1 Pint,	pt.
2 Pints	"	1 Quart,	qt.
4 Quarts	"	1 Gallon,	gal.

gal.		qt.		pt.		gi.
				1	=	4
		1	=	2	=	8
1	=	4	=	8	=	32

NOTE 1.—This table is used in measuring all liquids. The gallon is the unit

NOTE 2. — A gallon of oil, vinegar, cider, wine, brandy and other spirits, contains 231 cubic inches.

NOTE 3. — A gallon of milk, or of beer or other malt liquors, contains 282 cubic inches.

NOTE 4. — Vessels of various capacities from 50 to 150 or more gallons, are indiscriminately called hogsheads, pipes, butts, tuns, etc.

79. DRY MEASURE.

2 Pints (pt.)	make	1 Quart,	qt.
8 Quarts	"	1 Peck,	pk.
4 Pecks	"	1 Bushel,	bush.

				qt.		pt.
		pk.		1	=	2
bush.		1	=	8	=	16
1	=	4	=	32	=	64

NOTE 1. — This table is used in measuring grain, fruit, potatoes, charcoal, etc.

NOTE 2. — A bushel is equal to $2150\frac{1}{2}$ cubic inches, nearly.

80. TIME.

60 Seconds (sec.)	make	1 Minute,	m.
60 Minutes	"	1 Hour,	h.
24 Hours	"	1 Day,	d.
7 Days	"	1 Week,	wk.
4 Weeks	"	1 Lunar Month,	l. m.
13 Months, 1 Day and 6 Hours	"	1 Julian Year,	J. yr.
12 Calendar Months (= 365 or 366 Days),		1 Civil Year,	c. yr.

										m.		sec.
								h.		1	=	60
						d.		1	=	60	=	3600
				wk.		1	=	24	=	1440	=	86400
		l. m.		1	=	7	=	168	=	10080	=	604800
J. yr.		1	=	4	=	28	=	672	=	40320	=	2419200
1	=	$13\frac{5}{112}$	=	$52\frac{5}{28}$	=	$365\frac{1}{4}$	=	8766	=	525960	=	31557600

(a) The twelve calendar months have the following number of days: January (Jan.) has 31 days; February (Feb.), 28; March (Mar.), 31; April (Apr.), 30; May, 31; June, 30; July, 31; August (Aug.), 31; September (Sept.), 30; October (Oct.), 31; November (Nov.), 30; December (Dec.), 31.

NOTE. — In leap year February has 29 days.

(b) The number of days in each month may be easily remembered by committing the following lines: —

"Thirty days hath September,
April, June and November;
All the rest have thirty-one,
Save the second month alone,
Which has just eight and a score
Till leap year gives it one more."

81. CIRCULAR AND ASTRONOMICAL MEASURE.

60 Seconds (60″)	make	1 Minute,	1′
60 Minutes	"	1 Degree,	1°
30 Degrees	"	1 Sign,	s.
12 Signs	"	1 Circumference,	circ.

						1′	=	60″
		s.		1°	=	60	=	3600
circ.		1	=	30	=	1800	=	108000
1	=	12	=	360	=	21600	=	1296000

NOTE. — This table is much used in trigonometrical and astronomical calculations.

82. MISCELLANEOUS TABLE.

12 Single Things	make	1 Dozen.
12 Dozen	"	1 Gross.
12 Gross	"	1 Great Gross.
20 Single Things	"	1 Score.
112 Pounds	"	1 Quintal of Fish.
196 Pounds	"	1 Barrel of Flour.
200 Pounds	"	1 Barrel of Beef or Pork.
24 Sheets	"	1 Quire of Paper.
20 Quires	"	1 Ream.

NOTE. — This table may be extended indefinitely.

83. A sheet folded in 2 leaves is called a folio.

"	"	"	" 4	"	"	"	a quarto or 4to.
"	"	"	" 8	"	"	"	an octavo or 8vo.
"	"	"	" 12	"	"	"	a duodecimo or 12mo
"	"	"	" 18	"	"	"	an 18mo.
"	"	"	" 24	"	"	"	a 24mo.

§ 8. REDUCTION.

84. Reduction is changing numbers of one denomination to those of another, *without changing their value.**

It is of two kinds, viz., *Reduction Descending* and *Reduction Ascending*.

85. *Reduction Descending* consists in changing numbers from a *higher* to a *lower* denomination.

86. *Reduction Ascending* is changing numbers from a *lower* to a *higher* denomination.

87. Reduction Descending is performed by *multiplication;* thus, to reduce 15£ to shillings, we multiply 15 by 20, because there will be 20 times as many shillings as pounds. So to reduce 15£ and 12s. to shillings, we multiply 15 by 20, and to the product add the 12s.

In a similar manner all such examples are reduced. Hence,

To reduce the higher denominations of a compound number to a lower denomination,

Rule.—*Multiply the highest denomination given, by the number it takes of the next lower denomination to make one of this higher, and to the product add the number of the lower denomination; multiply this sum by the number it takes of the* NEXT *lower denomination to make one of* THIS; *add as before, and so proceed till the number is brought to the denomination required.*

Ex. 1. Reduce 11£ 17s. 9d. 3qr. to farthings.

OPERATION.

```
 11. 17. 9. 3.
  20
 ----
 237 s.
  12
 ----
2853 d.
   4
 ----
11415 qr., Ans.
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Eleven pounds = 220s., and the 17s. added, make 237s. = 2844d., and the 9d. added, give 2853d. = 11412qr., which, increased by the 3qr., give 11415 qr., the answer.

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* Reduction, with merchants, means a diminution of the price of an article.

2. Reduce 7 l.m. 3wk. 5d. 19h. 53m. 45sec. to seconds.
Ans. 19252425 sec.
3. Reduce 53bush. 3pk. 7qt. 1pt. to pints.
4. Reduce 4c. yd. 24c. ft. 1695c. in. to inches.
5. Reduce 67t. 17cwt. 2qr. 23 lb. 14oz. 11dr. to drams.
6. Reduce 17lb. 10℥. 7ʒ. 2℈. 19gr. to grains.
7. Reduce 273lb. 11oz. 19dwt. 21gr. to grains.
8. Reduce 87yd. 3qr. 2na. to nails.
9. Reduce 69m. 5fur. 37rd. to rods.
10. Reduce 5 yd. 2ft. 7in. 2b. c. to barleycorns.
11. Reduce 7m. 7fur. 9ch. 3rd. 21li. to links.
12. Reduce 37sq. m. 637a. 3r. 37sq.rd. to square rods.
13. Reduce 14c. 7c. ft. 15cu. ft. 1716 cu. in. to cubic inches.
14. Reduce 15gal. 2qt. 1pt. 3gi. to gills.
15. Reduce 14circ. 7s. 17° 57′ 14″ to seconds.
16. Reduce 6 l.m. 2wk. 18h. 47 sec. to seconds.
17. Reduce 4m. 8ch. 2rd. to links.
18. Reduce 14t. 24lb. to ounces.

88. Reduction Ascending is performed by *division*, thus,

Ex. 1. To reduce 11415 farthings to pence, we divide the 11415 by 4, because there will be only one-fourth as many pence as farthings. Performing the division we obtain 2853d. and a remainder of 3qr. If we wish to reduce the 2853d. to shillings, we divide by 12, because there will be only one-twelfth as many shillings as pence, and obtain 237s. and a remainder of 9d. Again the 237s. may be reduced to pounds, by dividing by 20, giving 11£ and a remainder of 17s. Thus we find that 11415qr. are equal to 11£ 17s. 9d. and 3qr. Like reasoning applies to all similar examples. Hence,

To reduce a number of a lower denomination to numbers of higher denominations,

Rule.—*Divide the given number by the number it takes of that denomination to make one of the next higher; divide the quotient by the number it takes of* THAT *denomination to make*

one of the NEXT *higher, and so proceed till the number is brought to the denomination required. The last quotient, together with the several remainders* (50) *will be the answer.*

89. Reduction Ascending and Reduction Descending *prove each other.*

Ex. 2. Reduce 19252425 seconds to numbers of higher denominations. Ans. 7 l.m. 3wk. 5d. 19h. 53m. 45sec.

OPERATION.

60) 19252425
60) 320873 . . . 45sec.
24 = 8 × 3. 8) 5347 . . . 53m.
3) 668 . . . 3
7) 222 . . . 2 . . . 3 + 2 × 8 = 19h.
4) 31 . . . 5d.
l. m. 7 . . . 3 wk.

First divide by 60 (58, b) to reduce the seconds to minutes, then divide by 60 to reduce minutes to hours; then by 24, i. e. by 8 and 3 (56 and 57), to reduce hours to days; etc.

3. Reduce 3455 pints, dry measure, to higher denominations.
4. Reduce 229791 cubic inches to feet and yards.
5. In 34758123 drams, how many tons, cwt. etc.?
6. In 103199 grains, Apothecaries' weight, how many lbs. etc.?
7. In 1578237 grains Troy, how many lbs. etc.?
8. In 1406 nails how many yards?
9. Reduce 22317 rods to miles.
10. Reduce 635 barleycorns to yards.
11. Reduce 63996 links to miles.
12. Reduce 3890877 square rods to miles.
13. Reduce 3317748 cubic inches to cords.
14. How many gallons in 503 gills?
15. How many circumferences in 18964634 seconds?
16. How many lunar months in 15789647 seconds?

17. How many miles in 32850 links?

18. How many tons in 448384 ounces?

NOTE. — This subject will receive further attention in the sections on Fractions.

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§ 9. SIGNS, DEFINITIONS AND GENERAL PRINCIPLES.

90. The sign of *inequality*, $>$ or $<$, signifies that the number at the *opening* of the sign is greater than that at the *vertex;* thus, $5 + 3 > 7$, i. e. 5 and 3 are greater than 7. Again, $7 - 5 < 4$, i. e. 7 minus 5 is less than 4.

91. Parenthesis, ( ), indicates that all the numbers within it are to be subjected to the same operation; thus, $(8 + 4) \times 3 = 24 + 12 = 36$; also, $(15 - 6) \div 3 = 5 - 2 = 3$.

NOTE. — Without the parenthesis, the first example would stand thus: $8 + 4 \times 3 = 8 + 12 = 20$, i. e. the sign of multiplication would not affect the 8. So in the second example, if the parenthesis be removed, the sign of division will not affect the 15.

(a) A vinculum, $\overline{}$, placed over several numbers, performs the same office as the parenthesis, and, in any example where their aid is needed, *either* may be used; thus,

$$(8 + 5) \times 3 = \overline{8 + 5} \times 3 = 24 + 15 = 13 \times 3 = 39; \text{ also,}$$

$$(56 - 12) \div 4 = \overline{56 - 12} \div 4 = 14 - 3 = 44 \div 4 = 11.$$

NOTE. — The numbers in parenthesis or under a vinculum may be taken *separately*, or they may first be *united* and then the result may be multiplied or divided, as in the above examples.

92. ALL numbers are *even* or *odd.*

(a) An *even* number is divisible by 2 without a remainder; as 2, 4, 12, etc.

(b) An *odd* number is *not* divisible by 2 without remainder; as, 1, 3, 7, 15, etc.

93. ALL numbers are *concrete* or *abstract.*

(a) A number that is applied to a particular object, is *concrete;* as, 6 *books*, 11 *men*, 25 *horses*, 4 *bushels*, etc.

(b) A number *not* applied to any particular object, is *abstract;* as, 6, 11, 25, 4, etc.

94. ALL numbers are *prime* or *composite.*

(a) A *prime* number can be divided by no whole number except *itself* and *unity;* as, 1, 2, 3, 5, 7, 11, 19, etc.

NOTE 1.—*Two* is the only *even* prime number, *for all even numbers are divisible by* 2.

NOTE 2.—Two numbers are *mutually* prime when no whole number but one will divide each of them; thus, 7 and 11 are mutually prime; so, also, 9 and 16 are *mutually* prime, although neither 9 nor 16 is *absolutely* prime.

(b) A *composite* number (Art. 43) can be divided by other numbers besides itself and unity; as, $4 = 2 \times 2$, $6 = 2 \times 3$, $8 = 2 \times 4 = 2 \times 2 \times 2$, $15 = 3 \times 5$, etc.

NOTE 1.—A composite number which is composed of *any number of* EQUAL *factors* is called a *power*, and the *equal factors* are called the *roots* of the power, thus, 9, which equals 3×3, is the *second power* or *square* of 3, and 3 is the *second* or *square root* of 9; 27, which equals $3 \times 3 \times 3$, is the *third power* or *cube* of 3, and 3 is the *third* or *cube root* of 27; etc.

NOTE 2.—The *power* of a number is usually indicated by a figure, called an *index* or *exponent*, placed at the right and a little above the number; thus, the *second power* of 4 is written 4^2, which equals $4 \times 4 = 16$; the *third power* of 4 is 4^3, which equals $4 \times 4 \times 4 = 64$; etc.

NOTE 3.—A *root* is indicated by a *fractional exponent* or by the *radical sign*, $\sqrt{\ }$; thus, the *second* or *square root* of 9 is written $9^{\frac{1}{2}}$ or $\sqrt{9}$, either of which expressions is equal to 3, i. e. the *square root* of 9 is *one* of the *two equal factors* of 9; the *third* or *cube root* of 64 is $64^{\frac{1}{3}}$ or $\sqrt[3]{64}$, which equals 4, one of the *three* equal factors of 64; etc.

NOTE 4.—The third and higher roots *require a figure* over the radical sign.

NOTE 5.—*Every number is considered to be both the first power and the first root of itself.*

95. ALL composite numbers are *perfect* or *imperfect.*

(a) A *perfect* number is equal to one half the sum of all its integral factors; thus, 28 is a perfect number, for the sum of its factors, $1 + 2 + 4 + 7 + 14 + 28 = 56 = 2 \times 28$.

NOTE 1. — All the perfect numbers known are 1, 6, 28, 496, 8128, 33550336, 8589869056, 137438691328 and 2305843008139952128.

NOTE 2. — All known perfect numbers, except 1, end with 6 or 28.

(b) An *imperfect* number is *not* equal to one half the sum of its factors; thus, 8 is an imperfect number, for the sum of its factors, $1 + 2 + 4 + 8 = 15 < 2 \times 8$.

96. ALL imperfect numbers are *abundant* or *defective*.

(a) An *abundant* number is one the sum of whose factors is *greater* than twice the number; thus, 12 is an abundant number, for the sum of its factors, $1 + 2 + 3 + 4 + 6 + 12 = 28 > 2 \times 12$.

(b) A *defective* number is one the sum of whose factors is *less* than twice the number; thus, 16 is a defective number, for $1 + 2 + 4 + 8 + 16 = 31 < 2 \times 16$.

97. The *factors* of a number are those numbers whose continued product is the number; thus, 3 and 7 are the factors of 21; 3 and 6, or 3, 3 and 2 are the factors of 18; etc.

(a) The *prime* factors of a number are those *prime* numbers whose continued product is the number; thus, the prime factors of 24 are 2, 2, 2 and 3; the prime factors of 36 are 2, 2, 3 and 3; etc.

NOTE. — Since 1, as a factor, is useless, it is not here enumerated.

98. An *aliquot part* of a quantity is that which is contained in the quantity an exact number of times; thus $12\frac{1}{2}$ cents, 20 cents, 25 cents, $33\frac{1}{3}$ cents, etc. are aliquot parts of a dollar.

99. An *aliquant part* of a quantity is *not* contained in that quantity without remainder; thus, 12 cents, 23 cents, 75 cents, etc. are aliquant parts of a dollar.

100. A *measure* of any quantity is contained in that quantity a certain number of times without remainder; thus 3 is a measure of 6, and 8 of 24.

101. A *multiple* of any quantity contains that quantity a certain number of times without remainder; thus, 15 is a multiple of 5, and 21 of 3.

NOTE 1.—Measure and multiple are *correlative terms;* i. e., if one number is a measure of another, the 2d is *necessarily* a multiple of the 1st; thus, if 3 is a measure of 12, then 12 is a multiple of 3.

NOTE 2.—A measure is sometimes called a *sub-multiple;* thus, 3 is a submultiple of 12.

NOTE 3.—*Every number is considered both a measure and a multiple of itself.*

102. A *common measure* of two or more numbers is *any number that will divide each of them without remainder;* thus, 3 is a common measure of 12, 18 and 30.

103. The *greatest common measure* of two or more numbers is the *greatest* number that will measure each of them; thus, 6 is the greatest common measure of 12, 18 and 30.

104. A *common multiple* of two or more numbers is *any number that may be divided by each of them without remainder;* thus, 48 is a common multiple of 4, 6 and 8.

105. The *least common multiple* of two or more numbers is the *least* number that is exactly divisible by each of the given numbers; thus, 24 is the least common multiple of 4, 6 and 8.

NOTE 1.—Least common measure, and greatest common multiple, are absurd expressions.

NOTE 2.—The terms divisor, factor, measure, sub-multiple, component part, and aliquot part, have a kindred signification.

106. The *reciprocal* of a number is the quotient which arises from dividing a unit by that number; thus, the reciprocals of 4, 9 and 12 are $\frac{1}{4}$, $\frac{1}{9}$ and $\frac{1}{12}$.

PROBLEM 1.

107. To find all the prime numbers from 1 up to any given number,

RULE.—*Write the odd numbers in the order of their values, beginning with* 1; *thus,* 1, 3, 5, 7, 9, *etc. Write* 3 *under every*

3d number after 3, 5 under every 5th after 5, 7 under every 7th after 7, etc. Those numbers having no figures placed under them, together with 2, will be all the prime numbers, so far as the table is extended.

Example. What are the prime numbers from 1 to 45?

1,	3,	5,	7,	9,	11,	13,	15,	17,	19,	21,	23,
				3			3.5			3.7	
25,	27,	29,	31,	33,	35,	37,	39,	41,	43,	45.	
5	3.9			3 11	5.7		3.13			15.3,5.9	

Ans. 1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41 and 43.

The reason of this rule is obvious. The 3d number from 3 is formed by adding 2 to 3, 3 times, i. e., by adding twice 3 to 3; hence the sum, 9, is divisible by 3, and is ∴ not prime; also, the 3d number from 9 is formed by adding twice 3 to 9, and so on; hence every 3d number from 3 is composite. In like manner, the 5th number from 5 is formed by adding twice 5 to 5, and is ∴ divisible by 5, and also by 3, etc., etc.

(a) Not only does this process point out the prime numbers, but it also gives the factors of the intervening odd composite numbers; thus, under 9 is 3, one of the factors of 9; under 15 are 3 and 5, factors of 15; under 45 are 15 and 3, also 5 and 9, i. e., two sets of factors of 45, and if either of these sets (e. g. $15 \times 3 = 5 \times 3 \times 3$) be factored, we have *all the prime factors of* 45.

Note 1.—This mode of finding the prime numbers was invented by Eratosthenes, a Cyrenian Greek, an eminent mathematician in the times of the Ptolemies, who, having written the odd numbers as above, *cut out* every 3d number from 3, every 5th from 5, etc., and, as this gave to his sheet a strong resemblance to a common sieve, it was called "the sieve of Eratosthenes."

Note 2.—The process for finding prime numbers becomes very tedious when carried to a great extent; hence, analysts have sought a general formula for the purpose, but their efforts, hitherto, have been fruitless.

TABLE OF PRIME NUMBERS FROM 1 TO 3407.

1	173	409	659	941	1223	1511	1811	2129	2423	2741	3079
2	179	419	661	947	1229	1523	1823	2131	2437	2749	3083
3	181	421	673	953	1231	1531	1831	2137	2441	2753	3089
5	191	431	677	967	1237	1543	1847	2141	2447	2767	3109
7	193	433	683	971	1249	1549	1861	2143	2459	2777	3119
11	197	439	691	977	1259	1553	1867	2153	2467	2789	3121
13	199	443	701	983	1277	1559	1871	2161	2473	2791	3137
17	211	449	709	991	1279	1567	1873	2179	2477	2797	3163
19	223	457	719	997	1283	1571	1877	2203	2503	2801	3167
23	227	461	727	1009	1289	1579	1879	2207	2521	2803	3169
29	229	463	733	1013	1291	1583	1889	2213	2531	2819	3181
31	233	467	739	1019	1297	1597	1901	2221	2539	2833	3187
37	239	479	743	1021	1301	1601	1907	2237	2543	2837	3191
41	241	487	751	1031	1303	1607	1913	2239	2549	2843	3203
43	251	491	757	1033	1307	1609	1931	2243	2551	2851	3209
47	257	499	761	1039	1319	1613	1933	2251	2557	2857	3217
53	263	503	769	1049	1321	1619	1949	2267	2579	2861	3221
59	269	509	773	1051	1327	1621	1951	2269	2591	2879	3229
61	271	521	787	1061	1361	1627	1973	2273	2593	2887	3251
67	277	523	797	1063	1367	1637	1979	2281	2609	2897	3253
71	281	541	809	1069	1373	1657	1987	2287	2617	2903	3257
73	283	547	811	1087	1381	1663	1993	2293	2621	2909	3259
79	293	557	821	1091	1399	1667	1997	2297	2633	2917	3271
83	307	563	823	1093	1409	1669	1999	2309	2647	2927	3299
89	311	569	827	1097	1423	1693	2003	2311	2657	2939	3301
97	313	571	829	1103	1427	1697	2011	2333	2659	2953	3307
101	317	577	839	1109	1429	1699	2017	2339	2663	2957	3313
103	331	587	853	1117	1433	1709	2027	2341	2671	2963	3319
107	337	593	857	1123	1439	1721	2029	2347	2677	2969	3323
109	347	599	859	1129	1447	1723	2039	2351	2683	2971	3329
113	349	601	863	1151	1451	1733	2053	2357	2687	2999	3331
127	353	607	877	1153	1453	1741	2063	2371	2689	3001	3343
131	359	613	881	1163	1459	1747	2069	2377	2693	3011	3347
137	367	617	883	1171	1471	1753	2081	2381	2699	3019	3359
139	373	619	887	1181	1481	1759	2083	2383	2707	3023	3361
149	379	631	907	1187	1483	1777	2087	2389	2711	3037	3371
151	383	641	911	1193	1487	1783	2089	2393	2713	3041	3373
157	389	643	919	1201	1489	1787	2099	2399	2719	3049	3389
163	397	647	929	1213	1493	1789	2111	2411	2729	3061	3391
167	401	653	937	1217	1499	1801	2113	2417	2731	3067	3407

Problem 2.

108. To resolve a number into its prime factors,

Rule.—*Divide the given number by any prime number greater than one, that will divide it; divide the quotient as before, and so on till the quotient is prime. The several divisors and last quotient will be the prime factors sought.*

Ex. 1. What are the prime factors of 30? Ans. 2, 3 and 5.

OPERATION.

$$\begin{array}{r|l} 2 & 30 \\ \hline 3 & 15 \\ \hline & 5 \end{array}$$

It is immaterial in what order the prime factors are taken, though it will usually be most convenient to take the smaller factors first.

2. Resolve 96 into its prime factors. Ans. 2, 2, 2, 2, 2 and 3.
3. Resolve 180 into its prime factors. Ans. 2, 2, 3, 3 and 5.
4. What are the prime factors of 7684? Ans. 2, 2, 17 and 113.
5. What are the prime factors of 242550?
6. What are the prime factors of 1801800?

Problem 3.

109. To find *all* the integral factors of a number,

Rule.—1. *Resolve the number into its prime factors.*

2. *Form a series* of numbers by writing* 1 *and the* 1*st*, 2*d*, *etc., powers* (94, b, Note 1) *of some one of the prime factors up to the highest power of that factor contained in the given number. Do the same with each of the different prime factors.*

3. *Multiply the terms of one of these series by the terms of another, term by term, and keep the terms of the product separate; then multiply the terms of this product by the terms of another series, and so on until each series has been employed. The terms*

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* Three or more numbers in succession, such that each succeeding number is formed from one or more of the preceding, in accordance with some fixed law, constitute *a series*. The several numbers forming the series are the *terms* of the series.

of the last product will be all the divisors of the given number.

Ex. 1. Find all the integral factors of 36.

$36 = 2^2 \times 3^2$

1st series $= 1,\ 2,\ 2^2 = 1, 2, 4$
2d series $= 1,\ 3,\ 3^2 = 1, 3, 9$

Product of the two series = 1, 2, 4, 3, 6, 12, 9, 18, 36, Ans.

The truth of the rule is obvious. All the prime factors of the number are taken, and they are also multiplied together in every possible way, *two* factors at a time, *three* at a time, etc., and hence every possible factor of the number is found.

2. Find the divisors of 1800.

$1800 = 2^3 \times 3^2 \times 5^2.$

1st series = 1, 2, 4, 8
2d series = 1, 3, 9

Prod. of 1st and 2d series = 1, 2, 4, 8, 3, 6, 12, 24, 9, 18, 36, 72
3d series = 1, 5, 25

Continued Prod. of 1st, 2d and 3d series = 1, 2, 4, 8, 3, 6, 12, 24, 9, 18, 36, 72, 5, 10, 20, 40, 15, 30, 60, 120, 45, 90, 180, 360, 25, 50, 100, 200, 75, 150, 300, 600, 225, 450, 900, 1800, Ans.

Note. — This rule is very convenient in reducing the higher equations in Algebra.

3. What are the divisors of 144?

4. What are the divisors of 500?

Problem 4.

110. To find the greatest common measure of two or more numbers,

Rule 1. — *Resolve each number into its prime factors, and the continued product of all the prime factors that are common to all the given numbers will be the common measure sought.*

Ex. 1. What is the greatest common measure of 18, 30 and 48? Ans. $2 \times 3 = 6$.

OPERATION.

$$18 = 2 \times 3 \times 3$$
$$30 = 2 \times 3 \times 5$$
$$48 = 2 \times 3 \times 2 \times 2 \times 2$$

We see that 2 and 3 are factors *common* to all the numbers, and, furthermore, they are the *only* common factors; hence their product, $2 \times 3 = 6$, is the greatest common measure of the given numbers.

2. What is the greatest common measure of 60, 72, 48 and 84? Ans. $2^2 \times 3 = 12$.

OPERATION.

$$60 = 2 \times 2 \times 3 \times 5$$
$$72 = 2 \times 2 \times 2 \times 3 \times 3$$
$$48 = 2 \times 2 \times 2 \times 2 \times 3$$
$$84 = 2 \times 2 \times 3 \times 7$$

Although 2 is a factor *more than twice* in *some* of the given numbers, yet, as it is a factor *only* twice in *others*, we are at liberty to take 2 *but* twice in finding the greatest common measure. The same remark applies to other factors.

3. What is the greatest common measure of 30, 90, 120, 210 and 60? Ans. $2 \times 3 \times 5 = 30$.

4. Find the greatest common measure of 25, 75, 90, 85, 100 65, 125 and 250. Ans. 5.

5. Find the greatest common measure of 42, 63, 105, 147, 189 and 168.

6. Find the greatest common measure of 72, 120, 144, 168 and 48.

(a) When the given numbers are not readily resolved into their prime factors, their greatest common measure may be more easily found by

RULE 2. — *Divide the greater of two numbers by the less, and, if there be a remainder, divide the divisor by the remainder, and continue dividing the last divisor by the last remainder until nothing remains; the last divisor is the greatest common measure of the two numbers.*

If more than two numbers are given, find the greatest measure of two of them, then of this measure and a third number, and so on until all the numbers have been taken ; the last divisor will be the measure sought.

7. What is the greatest common measure of 14 and 20 ?

Ans. 2.

OPERATION.

$$\begin{array}{l} 14)\,20\,(1 \\ \quad\;\;\underline{14} \\ \qquad\;\; 6)\,14\,(2 \\ \qquad\qquad\;\underline{12} \\ \qquad\qquad\quad 2)\,6\,(3 \\ \qquad\qquad\qquad\;\underline{6} \\ \qquad\qquad\qquad\;0 \end{array}$$

111. Before explaining this operation, *four* principles may be stated, viz.: —

(a) Every number is a measure of itself (101, Note 3).

(b) If one number measures another, the 1st will measure any multiple of the 2d ; thus, if 3 measures 12 it will measure 5 times 12, or any number of times 12.

(c) If a number measures each of two numbers, it will measure their *sum* and also their *difference ;* thus, since 6 is contained in 30 *five* times, and in 12 *twice*, in $30 + 12 = 42$, it will be contained $5 + 2 = 7$ times, and in $30 - 12 = 18$, it will be contained $5 - 2 = 3$ times.

(d) Not only will the greatest common measure of two numbers measure their difference, but, unless one of the numbers is a multiple of the other, it will also measure the remainder, after one of the numbers has been taken from the other, as many times as possible; thus, the greatest measure of 6 and 22 will measure $22 - 3 \times 6 = 4$.

112. It may now be shown, 1st, that 2 is *a common measure* of 14 and 20, and, 2d, that it is their *greatest* common measure.

1st. 2 measures 6, ∴ (111, b) 2 measures $6 \times 2 = 12$, and (111, c) 2 measures $2 + 12 = 14$; again, since 2 measures 6

and 14 (111, c) it measures $6 + 14 = 20$; i. e. 2 measures 14 and 20.

2d. The greatest measure of 14 and 20 (111, c) must measure $20 - 14 = 6$, ∴ it cannot be greater than 6; again, the greatest measure of 6 and 14 (111, d) must measure $14 - 6 \times 2 = 2$, ∴ the greatest common measure of 14 and 20 cannot exceed 2, and, as it has been previously shown that 2 is a measure of 14 and 20, *it is their greatest measure.*

A similar explanation is applicable in all cases.

113. It will be seen that, in finding the common measure of 14 and 20, we are led to find the measure of 6 and 14, then of 2 and 6; i. e. in any example, we seek to find the measure of the remainder and divisor, then of the *next* remainder and divisor, and so on, until the greatest measure of the last remainder, and the divisor which gave that remainder, is found, and this measure will be the greatest common measure of the two given numbers; thus, the question becomes more and more simple as each successive step is taken in the operation.

8. What is the greatest common measure of 27088 and 39912?
Ans. 8.

9. Find the greatest common measure of 437437 and 2018835.
Ans. 91.

10. Find the greatest common measure of 16, 24 and 36.
Ans. 4.

```
        FIRST OPERATION.                    SECOND OPERATION.
        16)24(1                             24)36(1
           16                                  24
           --                                  --
            8)  6(2                            12)24(2
               16                                 24
               --                                 --
                0                                  0

Again,   8)36(4               Again,   12)16(1
           32                             12
           --                             --
            4)8(2                          4)12(3
              8                              12
              -                              --
              0                               0
```

First find the measure of 16 and 24, viz., 8, and then find the measure of 8 and 36; or, first find the measure of 24 and 36, viz., 12, and then of 12 and 16; or, we might first find the measure of 16 and 36, and then of that measure and 24.

11. Find the greatest common measure of 9360, 437437 and 2018835. Ans. 13.

12. Find the greatest common measure of 1269729, 405405 and 5816907.

13. What is the greatest common measure of 8 and 15?

Ans. 1.

14. What is the greatest common measure of 8, 12 and 33?

15. Find the greatest common measure of 1181, 2741 and 3413.

Problem 5.

114. To find the least common multiple of two or more numbers,

Rule 1.—*Resolve each number into its prime factors, and the continued product of the highest powers of all the different prime factors contained in the given numbers, will be the multiple sought.*

Ex. 1. What is the least common multiple of 24, 36 and 20?

Ans. $2 \times 2 \times 2 \times 3 \times 3 \times 5 = 2^3 \times 3^2 \times 5 = 360$.

OPERATION.

$24 = 2 \times 2 \times 2 \times 3$
$36 = 2 \times 2 \times 3 \times 3$
$20 = 2 \times 2 \times 5$

Since 360 contains all the factors of 24, 36 and 20, respectively, it, evidently, is divisible by each of those numbers. It, also, is evident that no number less than 360 will contain 24, 36 and 20, for if one of the 2's in the common multiple were omitted, it would not contain 24; if one of the 3's, it would not contain 36; and if the 5 were omitted, it would not contain 20.

2. What is the least common multiple of 6, 8, 12, 18 and 24?

Ans. $2^3 \times 3^2 = 72$.

3. Find the least common multiple of 48, 96, 144 and 192.

4. Find the least common multiple of 33, 44 and 55.

5. Find the least common multiple of 3, 8, 27, 24, 54, 48, 90 and 45.

6. Find the least common multiple of 18, 28, 56, 64 and 72.

(a) The above rule is always applicable, but the same end may sometimes be more easily attained by

RULE 2.—*Having set the given numbers in a line, divide by any* PRIME *number that will divide two or more of them, and set the quotients and undivided numbers in a line beneath; proceed with this line as with the first, and so continue until no two of the numbers can be divided by any number greater than one; the continued product of the divisors and numbers in the last line will be the multiple sought.*

This rule may be illustrated by the example already employed in explaining the first rule, viz., What is the least common multiple of 24, 36 and 20?

Ans. $2 \times 2 \times 3 \times 2 \times 3 \times 5 = 360$.

OPERATION.

$$\begin{array}{r|rrr} 2) & 24, & 36, & 20 \\ \hline 2) & 12, & 18, & 10 \\ \hline 3) & 6, & 9, & 5 \\ \hline & 2, & 3, & 5 \end{array}$$

If the process by the 1st rule be examined it will be seen that the factor 2 is found 7 times in the given numbers, and as 2 is taken but 3 times in finding the multiple, it is rejected 4 times. By the 2d rule, also, 2 is rejected 4 times, viz., twice in the 1st division by 2 and twice in the 2d division by 2. The learner may think 2 is rejected 3 times in each of the two first divisions, but he must remember that the *divisor*, 2, *is retained* as a factor in the common multiple in each instance.

Similar remarks are applicable to all rejected factors in like examples, ∴ the two rules give identical results.

NOTE.—The principle, which is the same in the two rules, is most readily perceived by the first operation.

7. What is the least common multiple of 5, 16, 24, 32 and 48?

Ans. $2^5 \times 3 \times 5 = 480$.

OPERATION.

By Rule 1.

$5 = 5$
$16 = 2 \times 2 \times 2 \times 2$
$24 = 2 \times 2 \times 2 \times 3$
$32 = 2 \times 2 \times 2 \times 2 \times 2$
$48 = 2 \times 2 \times 2 \times 2 \times 3$

By Rule 2.

$$\begin{array}{r|rrrrr}
2 & 5, & 16, & 24, & 32, & 48 \\ \hline
2 & 5, & 8, & 12, & 16, & 24 \\ \hline
2 & 5, & 4, & 6, & 8, & 12 \\ \hline
2 & 5, & 2, & 3, & 4, & 6 \\ \hline
3 & 5, & 1, & 3, & 2, & 3 \\ \hline
 & 5, & 1, & 1, & 2, & 1
\end{array}$$

8. What is the least common multiple of 6, 8, 12, 18, 24, 131 and 137? Ans. 1292184.

9. What is the least common multiple of 8, 15, 77 and 221?

10. What is the least common multiple of 10, 15, 45, 75 and 90?

(b) It is evident that 10, 15 and 45, in the above example, may at once be struck out; for each of these numbers is a measure of 90, and ∴ whatever multiple of 75 and 90 is found, *it*, certainly, must be a multiple of 10, 15 and 45; hence the question is reduced to this: What is the least common multiple of 75 and 90?

NOTE.—Many other abbreviations of this and other rules may be effected, but a delicate perception of the relations of numbers, and a skilful application of principles, will much more facilitate the progress of the learner than any set of formal rules.

11. What is the least common multiple of 4, 9, 6 and 8? Ans. 72.

12. What is the least common multiple of 8, 12, 16, 24, 32, 48 and 96? Ans. 96.

13. Find the least common multiple of 80, 20, 160, 40, 5, 320, 10 and 16.

14. Find the least common multiple of 91, 3523 and 6487. Ans. 12305839.

15. Find the least common multiple of 12089, 1309, 2849 and 2233.

16. Find the least common multiple of 28, 42, 56, 70, 80 and 90.

(c) If the numbers are prime, or even mutually prime, their product is their least common multiple.

17. What is the least common multiple of 8, 15 and 77?

Ans. 9240.

18. Find the least common multiple of 1181, 2741 and 3413.

(d) The least common multiple of *two* numbers is equal to their product divided by their greatest common measure.

19. What is the least common multiple of 12 and 20?

The greatest measure of 12 and 20 is 4, and

$12 \times 20 \div 4 = 60$, Ans.

20. What is the least common multiple of 35 and 49?

21. What is the least common multiple of 39 and 91?

115. The following facts will be found convenient in the subsequent rules:

(a) Every number whose unit figure is 0, or an *even* number, is itself *even*, and ∴ divisible by 2.

(b) If the *two* right hand figures of a number are divisible by 4, the whole number is also divisible by 4; e. g. $8724 = 8700 + 24$; now, 100 is divisible by 4, and ∴ 8700 is divisible by 4 (111, b); again, since 8700 and 24 is each divisible by 4, their sum, $8700 + 24 = 8724$, will also be divisible by 4 (111, c).

(c) Similar reasoning will show that the whole of a number is divisible by 8, if its last *three* figures are divisible by 8, etc., etc.

(d) A number ending with 5 or 0 is divisible by 5.

(e) Any number ending with 0 is divisible by 10.

(f) Every number the *sum* of whose digits is divisible by 9 is itself divisible by 9; e. g. $5643 = 5000 + 600 + 40 + 3$.

$$\begin{aligned}
\text{Now } 5000 &= 5 \times 1000 = 5 \times (999 + 1) = 5 \times 999 + 5, \\
600 &= 6 \times 100 = 6 \times (99 + 1) = 6 \times 99 + 6, \\
40 &= 4 \times 10 = 4 \times (9 + 1) = 4 \times 9 + 4;
\end{aligned}$$

$$\therefore 5643 = 5 \times 999 + 6 \times 99 + 4 \times 9 + 5 + 6 + 4 + 3;$$

again, it is evident that $5 \times 999 + 6 \times 99 + 4 \times 9$ is divisible by 9, and if $5 + 6 + 4 + 3$ ($= 18$) is divisible by 9, then the whole number, 5643, *must be* divisible by 9; but it will be seen that $5 + 6 + 4 + 3$ is the *sum* of the digits which express the number 5643; hence any number is divisible by 9 if the sum of its digits is divisible by 9.

If the sum of the digits of a number divided by 9 give a remainder, then the number itself divided by 9 *will give the* SAME *remainder.*

NOTE. — It is on these properties of the number 9 that the rules often given, for proving Addition, Subtraction, Multiplication and Division by casting out the 9's are founded.

(g) The properties given for 9 are equally true for 3; i. e. if the sum of the digits of a number is divisible by 3, the number is itself divisible by 3, and if the sum of the digits divided by 3 gives a remainder, then the number divided by 3 will give the *same* remainder.

(h) Any *even* number divisible by 3 is also divisible by 6; for, since it is *even*, it is divisible by 2, and, being divisible by 2 and by 3, it is divisible by $2 \times 3 = 6$.

NOTE. — The properties named in the foregoing paragraphs are dependent upon the given conditions; e. g. in (a), a number not ending in 0 or an even number is *not* divisible by 2; etc.

(i) Every prime number, except 2 and 5, must end with 1, 3 7 or 9; for,

1st. Every number must end with some one of the ten digits,	0, 1, 2, 3, 4, 5, 6, 7, 8, 9
2d. But no even number, except 2, is prime; ∴ take away	2, 4, 6, 8,
and we have remaining	0, 1, 3, 5, 7, 9,
3d. No number, except 5, ending in 0 or 5 can be prime,	0, 5,
∴ Every prime number, except 2 and 5, must end in	1, 3, 7 or 9.

NOTE. — The converse of this proposition is not true; i. e. a number ending with 1, 3, 7 or 9 *is not necessarily prime.*

§ 10. VULGAR FRACTIONS.

116. A FRACTION * is an expression representing one or more of the equal parts into which a unit is supposed to be divided.

117. *A Vulgar or Common Fraction* is expressed by two numbers, one above and the other below a line; thus, $\frac{1}{2}$ (one half), $\frac{2}{5}$ (two fifths), etc.

(a) The number below the line shows *into how many parts the unit is divided*, and is called the *denominator*, because it *denominates* or *gives name* to the parts; thus, if a unit is divided into 3 equal parts, each part is one *third;* if into 8, each part is one *eighth;* etc.

(b) The number above the line is called the *numerator*, because it *numerates* or *numbers* the parts *taken.*

(c) The numerator and the denominator are the *terms* of the fraction.

118. A fraction is nothing more nor less than *unexecuted division*, i. e. *division indicated but not performed*, the *numerator* being the *dividend* and the *denominator* the *divisor.* This is *the key* to a knowledge of fractions; and this knowledge of fractions is, in turn, the key to Higher Arithmetic and Algebra. *He who has the key intelligently in his possession will advance rapidly and pleasantly, while he who neglects the key will see no beauties in mathematics.*

(a) It follows from the above, that the value of a fraction is the quotient of the numerator divided by the denominator; thus, $\frac{12}{4} = 12 \div 4 = 3$.

119. A *proper fraction* is one whose numerator is *less* than the denominator; as, $\frac{2}{3}$, $\frac{7}{11}$, $\frac{8}{24}$, etc.

120. An *improper fraction* is one whose numerator *equals* or *exceeds* its denominator; as, $\frac{4}{4}$, $\frac{7}{7}$, $\frac{8}{5}$, $\frac{26}{13}$, etc. An improper

* *Fraction*, from the Latin *frango, to break.*

fraction *equals or exceeds a unit;* hence its *name* — IMPROPER *fraction.*

121. A *simple fraction* has but one numerator and one denominator, and is either *proper* or *improper;* as, $\frac{1}{3}$, $\frac{3}{5}$, $\frac{6}{6}$, $\frac{12}{7}$, etc.

122. A *compound fraction* is a fraction of a fraction; as, $\frac{2}{3}$ of $\frac{7}{11}$, $\frac{4}{2}$ of $\frac{3}{5}$ of $\frac{6}{9}$, etc.

123. A *mixed number* is a whole number and a fraction united; as, $3\frac{4}{5}$, $20\frac{3}{8}$, etc.

124. A *complex fraction* is one that has a fraction or a mixed number for one or for each of its terms; as, $\frac{3\frac{1}{4}}{7}$, $\frac{\frac{2}{5}}{6}$, $\frac{3}{2\frac{1}{2}}$, $\frac{7}{\frac{2}{4}}$, $\frac{8\frac{1}{3}}{7\frac{2}{6}}$, $\frac{\frac{3}{5}}{8\frac{3}{11}}$, etc.

REMARK. — The following are the most important operations in fractions.

CASE 1.

125. To multiply a fraction by a whole number,

RULE 1. — *Multiply the numerator by the whole number* (59, a, and 118); or,

RULE 2. — *Divide the denominator by the whole number* (59, d).

Ex. 1. Multiply $\frac{2}{15}$ by 3.

$\frac{2}{15} \times 3 = \frac{6}{15}$, by Rule 1; or,
$\frac{2}{15} \times 3 = \frac{2}{5}$, by Rule 2.

NOTE 1. — The 2d rule is preferable in this and all similar examples, because it gives the result in *smaller terms.*

2. Multiply $\frac{7}{12}$ by 3. Ans. $\frac{7}{4}$.
3. Multiply $\frac{3}{25}$ by 5. Ans. $\frac{3}{5}$.
4. Multiply $\frac{17}{299}$ by 13. Ans $\frac{17}{23}$.
5. Multiply $\frac{897}{16011}$ by 593.
6. Multiply $\frac{326}{16011}$ by 27.

7. Multiply $\frac{2}{11}$ by 3.

$$\tfrac{2}{11} \times 3 = \tfrac{6}{11}, \text{ by Rule 1; or,}$$
$$\tfrac{2}{11} \times 3 = \frac{2}{3\frac{2}{3}}, \text{ by Rule 2.}$$

NOTE 2. — The 1st rule is preferable in this and all similar examples, because the 2d gives a *complex fraction*.

8. Multiply $\frac{8}{15}$ by 2. Ans. $\frac{16}{15}$.
9. Multiply $\frac{19}{1667}$ by 15. Ans. $\frac{285}{1667}$.
10. Multiply $\frac{563}{3391}$ by 24.
11. Multiply $\frac{74}{210}$ by 143.
12. Multiply $\frac{9876}{35770}$ by 2717.
13. Multiply $\frac{2}{7}$ by 3.

(a) The correctness of the first rule is also apparent from the following reasoning: It is just as evident that 3 times $\frac{2}{7}$ are $\frac{6}{7}$ as that 3 times 2 cents are 6 cents; or that 3 times 2 are 6; i. e. when the numerator is multiplied by 3, the fraction represents 3 times as many parts as before, and each part remains of the same size; $\therefore$ *the fraction is multiplied by* 3.

(b) The 2d rule may be explained thus: 2 ninepences, in New England, are 25 cents; i. e. 2 times $\frac{1}{8}$ of a dollar are $\frac{1}{4}$ of a dollar, and, as evidently, 2 times $\frac{3}{8}$ are $\frac{3}{4}$; i. e. if the denominator is divided by 2, the fraction represents just as many parts as before, *but each part is twice as great*, and, $\therefore$, the whole fraction is twice as great.

14. Multiply $\frac{8}{25}$ by 15. $15 = 5 \times 3$.

$$\tfrac{8}{25} \times 5 = \tfrac{8}{5} \text{ and } \tfrac{8}{5} \times 3 = \tfrac{24}{5}, \text{ Ans.}$$

NOTE.—We may here, as in whole numbers (44), use the component parts of the multiplier, and, in using these component parts, we may apply the 1st or the 2d rule, or both.

15. Multiply $\frac{12}{78}$ by 66. $66 = 6 \times 11$.

$$\tfrac{12}{78} \times 6 = \tfrac{12}{13} \text{ and } \tfrac{12}{13} \times 11 = \tfrac{132}{13}, \text{ Ans.}$$

16. Multiply $\frac{23}{36}$ by 42. Ans. $\frac{161}{6}$.
17. Multiply $\frac{73}{108}$ by 84.
18. Multiply $\frac{156}{33}$ by 44.

19. Multiply $\frac{78569}{99}$ by 63.
20. Multiply $\frac{278}{1728}$ by 1008.
21. Multiply $\frac{4}{5}$ by 5. $\frac{4}{5} \times 5 = \frac{4}{1} = 4$, by Rule 2; ∴,

(c) If we multiply a fraction by its denominator, the product will be the numerator.

22. Multiply $\frac{18}{79}$ by 79. Ans. 18.
23. Multiply $\frac{174}{1957}$ by 1957.
24. Multiply $\frac{73}{59}$ by 59.
25. Multiply $\frac{74698}{5}$ by 5.

CASE 2.

126. To divide a fraction by a whole number,

RULE 1.—*Divide the numerator by the whole number* (59, b, and 118); or,

RULE 2.—*Multiply the denominator by the whole number* (59, c).

Ex. 1. Divide $\frac{20}{21}$ by 4.

$$\frac{20}{21} \div 4 = \frac{5}{21}, \text{ by Rule 1; or,}$$
$$\frac{20}{21} \div 4 = \frac{20}{84}, \text{ by Rule 2.}$$

NOTE 1.—The 1st rule is preferable in this example. Why?

2. Divide $\frac{27}{59}$ by 9. Ans. $\frac{3}{59}$.
3. Divide $\frac{84}{796}$ by 21.
4. Divide $\frac{125}{469}$ by 25.
5. Divide $\frac{91}{746}$ by 13.
6. Divide $\frac{16011}{1987}$ by 593.
7. Divide $\frac{3}{11}$ by 2.

$$\frac{3}{11} \div 2 = \frac{\frac{3}{2}}{11}, \text{ by Rule 1; or,}$$
$$\frac{3}{11} \div 2 = \frac{3}{22}, \text{ by Rule 2.}$$

NOTE 2.—The 2d rule is preferable in this example. Why?

8. Divide $\frac{8}{15}$ by 3. Ans. $\frac{8}{45}$.
9. Divide $\frac{87}{156}$ by 5.
10. Divide $\frac{1667}{756}$ by 894.

11. Divide $\frac{3391}{46}$ by 74.
12. Divide $\frac{47}{84}$ by 7695.
13. Divide $\frac{6}{7}$ by 3.

(a) It is just as evident that $\frac{1}{3}$ of $\frac{6}{7}$ is $\frac{2}{7}$, as that $\frac{1}{3}$ of 6 cents is 2 cents, or that $\frac{1}{3}$ of 6 is 2; i. e. the 1st rule may be explained by saying that, if the numerator is divided by 3, the fraction will express only $\frac{1}{3}$ as many parts, and each part remains of the same size; *hence the value of the fraction is divided by* 3.

(b) By the 2d rule each part expressed by the fraction is made smaller, while the number of parts taken remains the same; ∴ the value of the fraction is divided when we multiply the denominator.

14. Divide $\frac{8}{25}$ by 20. $20 = 4 \times 5$.

$$\tfrac{8}{25} \div 4 = \tfrac{2}{25} \text{ and } \tfrac{2}{25} \div 5 = \tfrac{2}{125}, \text{ Ans.}$$

NOTE.—See Art. 125, b, Note.

15. Divide $\frac{15}{73}$ by 35. $35 = 5 \times 7$.

$$\tfrac{15}{73} \div 5 = \tfrac{3}{73} \text{ and } \tfrac{3}{73} \div 7 = \tfrac{3}{511}, \text{ Ans.}$$

16. Divide $\frac{161}{6}$ by 42. Ans. $\frac{23}{36}$.
17. Divide $\frac{511}{9}$ by 84.
18. Divide $\frac{624}{3}$ by 44.
19. Divide $\frac{549983}{11}$ by 63.
20. Divide $\frac{1946}{12}$ by 1008.

CASE 3.

127. To multiply $\frac{2}{7}$ by $\frac{3}{5}$, 1st, $\frac{2}{7} \times 3 = \frac{6}{7}$ (125, Rule 1); but the multiplier, 3, is 5 times $\frac{3}{5}$, ∴ the product, $\frac{6}{7}$, is 5 times the product sought; hence, 2d, $\frac{6}{7} \div 5 = \frac{6}{35}$ (126, Rule 2) is the product sought; i. e.

$$\tfrac{2}{7} \times \tfrac{3}{5} = \tfrac{6}{35}. \quad \text{Hence,}$$

To multiply a fraction by a fraction,

RULE.—*Multiply the numerators together for a new numerator, and the denominators for a new denominator.*

Ex. 1. Multiply $\frac{4}{9}$ by $\frac{7}{15}$. Ans. $\frac{28}{135}$.
2. Multiply $\frac{17}{19}$ by $\frac{8}{13}$. Ans. $\frac{136}{247}$.
3. Multiply $\frac{23}{14}$ by $\frac{17}{29}$.
4. Multiply $\frac{18}{29}$ by $\frac{7}{143}$.
5. Multiply $\frac{23}{15}$ by $\frac{7}{5}$.

(a) To multiply by a fraction is only to multiply by the numerator, and then divide the product by the denominator. In Ex. 6 we multiply $\frac{12}{35}$ by 5 and obtain $\frac{12}{7}$ (125, Rule 2), and then $\frac{12}{7}$ divided by 6 gives $\frac{2}{7}$ (126, Rule 1), the result sought.

6. Multiply $\frac{12}{35}$ by $\frac{5}{6}$.

$$\frac{\overset{2}{\cancel{12}}}{\underset{7}{\cancel{35}}} \times \frac{\cancel{5}}{\cancel{6}} = \frac{2}{7}, \text{ Ans.}$$

Note 1.—In this simple operation is involved *the whole principle of canceling.*

7. Multiply $\frac{12}{35}$ by $\frac{14}{27}$.

$$\frac{\overset{4}{\cancel{12}}}{\underset{5}{\cancel{35}}} \times \frac{\overset{2}{\cancel{14}}}{\underset{9}{\cancel{27}}} = \frac{8}{45}, \text{ Ans.}$$

The 7th example is solved on the same principle as the 6th. Since the product of the numerators is a dividend, and that of the denominators a divisor (118), and since the quotient is not affected by dividing both dividend and divisor by the same number (61, Cor.), we may *cancel* (*strike out, or reject*) the factors 3 and 7 from both numerator and denominator; i. e. we may *divide* both numerator and denominator by 3 and 7, and thus obtain $\frac{8}{45}$, the product sought.

Note 2.—There can be no difficulty in canceling so long as we remember the simple *principle*, that it rests upon rejecting equal factors from dividend and divisor. The *process* is only to *strike out* or *cancel* the same factors from numerator and denominator, and it often saves much labor.

8. Multiply $\frac{46}{85}$ by $\frac{25}{23}$.

$$\frac{\overset{2}{\cancel{46}}}{\underset{17}{\cancel{85}}} \times \frac{\overset{5}{\cancel{25}}}{\cancel{23}} = \frac{10}{17}, \text{ Ans.}$$

9. Multiply $\frac{75}{94}$ by $\frac{8}{125}$. Ans. $\frac{12}{235}$.
10. Multiply $\frac{18}{35}$ by $\frac{21}{198}$. Ans. $\frac{3}{55}$.
11. Multiply $\frac{77}{108}$ by $\frac{18}{55}$.
12. Multiply $\frac{64}{315}$ by $\frac{75}{56}$.

(b) In canceling 3 and 5 in Example 13, we obtain the quotients 1 and 1 in the numerators, and whenever an entire term cancels we obtain 1 to place instead of the term canceled; but since 1, as a multiplier or divisor, is valueless, there is no need of retaining it under any circumstances except where all the numerators are canceled; in such a case, 1 is the true numerator and must be retained.

13. Multiply $\frac{3}{25}$ by $\frac{5}{12}$.

$$\frac{\overset{1}{\cancel{3}}}{\underset{5}{\cancel{25}}} \times \frac{\overset{1}{\cancel{5}}}{\underset{4}{\cancel{12}}} = \frac{1}{20}, \text{ Ans.}$$

14. Multiply $\frac{144}{365}$ by $\frac{73}{288}$.

$$\frac{\overset{1}{\cancel{144}}}{\underset{5}{\cancel{365}}} \times \frac{\overset{1}{\cancel{73}}}{\underset{2}{\cancel{288}}} = \frac{1}{10}, \text{ Ans.}$$

Note.—The 14th and similar examples may be more conveniently written as follows:—

$$\frac{\overset{1}{\cancel{144}} \times \overset{1}{\cancel{73}}}{\underset{5}{\cancel{365}} \times \underset{2}{\cancel{288}}} = \frac{1}{10}, \text{ Ans. as before.}$$

15. Multiply $\frac{25}{3}$ by $\frac{12}{5}$.

$$\frac{\overset{5}{\cancel{25}} \times \overset{4}{\cancel{12}}}{\cancel{3} \times \cancel{5}} = 20, \text{ Ans.}$$

16. Multiply $\frac{1728}{4465}$ by $\frac{893}{5184}$.
17. Multiply $\frac{698}{2429}$ by $\frac{347}{4886}$.
18. Multiply $\frac{976}{427}$ by $\frac{2135}{488}$. Ans. 10.
19. Multiply $\frac{84}{95}$ by $\frac{190}{21}$.
20. Multiply $\frac{876968}{476}$ by $\frac{5236}{109621}$.

(c) If $\frac{1}{4}$ of an apple be divided into 7 equal parts, one of those parts will be $\frac{1}{28}$ of the whole apple; and if $\frac{1}{7}$ of $\frac{1}{4}$ is $\frac{1}{28}$, then $\frac{1}{7}$ of $\frac{3}{4}$ will be $\frac{3}{28}$, and $\frac{5}{7}$ of $\frac{3}{4}$ will be $\frac{15}{28}$; i. e. the rule for reducing a compound fraction to a simple one is the same as that for multiplying a fraction by a fraction.

21. Multiply $\frac{3}{4}$ by $\frac{5}{7}$, i. e. reduce $\frac{3}{4}$ of $\frac{5}{7}$ to a simple fraction. Ans. $\frac{15}{28}$.
22. Reduce $\frac{8}{9}$ of $\frac{7}{11}$. Ans. $\frac{56}{99}$.
23. Reduce $\frac{4}{7}$ of $\frac{3}{5}$ of $\frac{2}{11}$. Ans. $\frac{24}{385}$.
24. Reduce $\frac{4}{5}$ of $\frac{5}{8}$ of $\frac{7}{9}$ of $\frac{3}{7}$.

$$\frac{\cancel{4}}{\cancel{5}} \times \frac{\cancel{5}}{\underset{2}{\cancel{8}}} \times \frac{\cancel{7}}{\underset{3}{\cancel{9}}} \times \frac{\overset{1}{\cancel{3}}}{\cancel{7}} = \frac{1}{6}, \text{ Ans.}$$

NOTE. — The principle of canceling can be profitably applied whenever the product of two or more numbers is to constitute a dividend, and the product of other numbers is to constitute a divisor, provided that there are equal factors in the dividend and divisor.

25. What is $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{4}{5}$ of $\frac{5}{6}$ of $\frac{6}{7}$ of $\frac{7}{8}$ of $\frac{8}{9}$?

$$\frac{1}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{3}}{\cancel{4}} \times \frac{\cancel{4}}{\cancel{5}} \times \frac{\cancel{5}}{\cancel{6}} \times \frac{\cancel{6}}{\cancel{7}} \times \frac{\cancel{7}}{\cancel{8}} \times \frac{\cancel{8}}{9} = \frac{1}{9}, \text{ Ans.}$$

26. What is $\frac{3}{5}$ of $\frac{10}{12}$ of $\frac{4}{9}$ of $\frac{7}{11}$ of $\frac{11}{21}$? Ans. $\frac{2}{27}$
27. What is $\frac{4}{9}$ of $\frac{45}{80}$ of $\frac{12}{35}$ of $\frac{28}{57}$ of $\frac{14}{49}$?
28. What is $\frac{55}{17}$ of $\frac{51}{99}$ of $\frac{9}{5}$ of $\frac{24}{8}$?
29. What is $\frac{8}{17}$ of $\frac{34}{9}$ of $\frac{81}{96}$ of $\frac{1728}{144}$?

(d) The multiplication of a *whole number by a fraction* is only a modification of the principle of the 3d case; for, if the whole number has 1 placed under it, it becomes a fraction without change of value; thus, $8 = \frac{8}{1}$, $17 = \frac{17}{1}$, etc.

30. Multiply 8 by $\frac{3}{5}$. $\frac{8}{1} \times \frac{3}{5}$, or $8 \times \frac{3}{5} = \frac{24}{5}$, Ans.
31. Multiply 35 by $\frac{3}{5}$. Ans. 21.

We may multiply by 3 and divide the product, 105, by 5; or, better, as it keeps the numbers smaller, we may divide by 5 and multiply the quotient, 7, by 3, and the result is the same, 21, by either process.

32. Multiply 48 by $\frac{5}{16}$. Ans. 15.
33. Multiply 64 by $\frac{7}{24}$. Ans. $\frac{56}{3}$.
34. Multiply 1056 by $\frac{746}{4224}$.

Case 4.

128. To divide $\frac{2}{3}$ by $\frac{5}{7}$, 1st, $\frac{2}{3} \div 5 = \frac{2}{15}$ (126, Rule 2); but the divisor, 5, is 7 times $\frac{5}{7}$, $\therefore$ (59, f) the quotient, $\frac{2}{15}$, is only $\frac{1}{7}$ of the quotient sought; hence, 2d, $\frac{2}{15} \times 7 = \frac{14}{15}$ (125, Rule 1) is the quotient sought; i. e.

$$\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \times \frac{7}{5} = \frac{14}{15}. \quad \text{Hence,}$$

To divide a fraction by a fraction,

Rule. — *Invert the divisor, and then proceed as in multiplication* (127).

Ex. 1. Divide $\frac{4}{7}$ by $\frac{9}{11}$. $\frac{4}{7} \div \frac{9}{11} = \frac{4}{7} \times \frac{11}{9} = \frac{44}{63}$, Ans.
2. Divide $\frac{8}{17}$ by $\frac{15}{11}$. Ans. $\frac{88}{255}$.
3. Divide $\frac{13}{16}$ by $\frac{29}{43}$.
4. Divide $\frac{91}{147}$ by $\frac{53}{13}$.

(a) The reciprocal of $\frac{5}{7}$ is $\frac{1}{\frac{5}{7}}$ (106), and, multiplying both numerator and denominator of this complex fraction, $\frac{1}{\frac{5}{7}}$, by 7, we obtain $\frac{7}{5}$; but multiplying both terms of a fraction by the same

number does not change its value (60, Cor.), $\therefore \dfrac{1}{\frac{5}{7}} = \frac{7}{5}$; i. e. the reciprocal of $\frac{5}{7}$ is $\frac{7}{5}$; and, *generally, the reciprocal of any fraction is that fraction inverted.*

$$\text{Again, } 12 \div 4 = 3, \text{ and } 12 \times \tfrac{1}{4} = 3;$$
$$\text{so, also, } \tfrac{5}{7} \div 4 = \tfrac{5}{28}, \text{ and } \tfrac{5}{7} \times \tfrac{1}{4} = \tfrac{5}{28};$$

hence $\frac{5}{7} \div 4 = \frac{5}{7} \times \frac{1}{4}$; i. e. *it matters not whether we divide by any number or multiply by its reciprocal.*

From the above, together with Art. 127, we have another explanation of the rule in Art. 128.

5. Divide $\frac{19}{25}$ by $\frac{17}{49}$.
6. Divide $\frac{83}{57}$ by $\frac{79}{143}$.
7. Divide $\frac{3}{4}$ of $\frac{4}{5}$ by $\frac{2}{7}$ of $\frac{7}{11}$.

$$\frac{3}{4} \times \frac{4}{5} \div \frac{2}{7} \times \frac{7}{11} = \frac{3}{\cancel{4}} \times \frac{\cancel{4}}{5} \times \frac{\cancel{7}}{2} \times \frac{11}{\cancel{7}} = \frac{33}{10}, \text{ Ans}$$

8. Divide $\frac{7}{8}$ of $\frac{16}{19}$ by $\frac{4}{11}$ of $\frac{33}{29}$. Ans. $\frac{203}{114}$.
9. Divide $\frac{6}{11}$ of $\frac{5}{8}$ of $\frac{3}{15}$ by $\frac{4}{7}$ of $\frac{9}{11}$.
10. Divide $\frac{16}{48}$ of $\frac{7}{15}$ by $\frac{11}{13}$ of $\frac{9}{4}$ of $\frac{18}{27}$.

(b) If the denominator of the divisor is like that of the dividend, as in Ex. 11, they may both be disregarded; for, evidently, $\frac{6}{27}$ is contained in $\frac{24}{27}$ just as many times as 6 apples are contained in 24 apples, or 6 in 24; i. e. $\frac{24}{27} \div \frac{6}{27} = 24 \div 6 =$ numerator of dividend $\div$ numerator of divisor; and this is equally true when the numerator of the dividend is not a multiple of the numerator of the divisor; thus, $\frac{5}{\cancel{7}} \div \frac{3}{\cancel{7}} = 5 \div 3 = \frac{5}{3}$.

11. Divide $\frac{24}{27}$ by $\frac{6}{27}$. Ans. 4.
12. Divide $\frac{33}{57}$ by $\frac{11}{57}$. Ans. 3.
13. Divide $\frac{28}{17}$ by $\frac{21}{17}$. Ans. $\frac{28}{21}$.
14. Divide $\frac{73}{99}$ by $\frac{83}{99}$. Ans. $\frac{73}{83}$.
15. Divide $\frac{46}{95}$ by $\frac{2}{95}$.
16. Divide $\frac{49}{78}$ by $\frac{28}{78}$.
17. Divide $\frac{53}{1927}$ by $\frac{17}{1927}$.

(c) When the numerator and denominator of the divisor are respectively factors of the corresponding terms of the dividend, as in Ex. 18, it is best to divide numerator by numerator, and denominator by denominator. This mode is *true* in *all* fractions, but *not always convenient.* Why true? Why not convenient?

18. Divide $\frac{36}{77}$ by $\frac{4}{7}$. Ans. $\frac{9}{11}$.
19. Divide $\frac{964}{1563}$ by $\frac{2}{3}$.
20. Divide $\frac{96}{1728}$ by $\frac{24}{144}$.

Case 5.

129. To reduce a fraction to its smallest or lowest terms,

Rule 1.—*Divide each term by any factor common to them, then divide these quotients by any factor common to* THEM, *and so proceed till the quotients are mutually prime* (61, Cor.); or,

Rule 2.—*Divide each term by their greatest common measure* (110).

Ex. 1. Reduce $\frac{420}{700}$ to its lowest terms.

$$7\,\big]\;10\,\big]\;\frac{42.0}{70.0}\Big|\overset{2}{\frac{6}{10}}\Big|\frac{3}{5},\text{ Ans. by Rule 1.}$$

$$140\,\big]\;\frac{420}{700}\Big|\frac{3}{5},\text{ Ans. by Rule 2.}$$

In the first operation, we divide by 10 by cutting off 0 in each term (58), then divide by 7, then by 2.

2. Reduce $\frac{44}{96}$ to its lowest terms. Ans. $\frac{11}{24}$.
3. Reduce $\frac{3675}{6615}$. Ans. $\frac{5}{9}$.
4. Reduce $\frac{37297}{90649}$.
5. Reduce $\frac{2401329}{2403051}$.
6. Reduce $\frac{17955}{34965}$.
7. Reduce $\frac{360}{3024}$.
8. Reduce $\frac{17955}{84965}$.

9. Reduce $\frac{210}{6300}$.
10. Reduce $\frac{150}{3640}$.
11. Reduce $\frac{11700}{10400}$.
12. Reduce $\frac{2394}{28723}$.
13. Reduce $\frac{62328}{44520}$.
14. Reduce $\frac{3600}{225}$.
15. Reduce $\frac{308448}{151113952}$.

Case 6.

130. How many units in $\frac{13}{4}$?

One unit $= \frac{4}{4}$, and $\therefore$ $\frac{13}{4}$ will be reduced to units by dividing $\frac{13}{4}$ by $\frac{4}{4}$; thus (128, b), $\frac{13}{4} \div \frac{4}{4} = 13 \div 4$ ($=$ numerator $\div$ denominator) $= 3\frac{1}{4}$. Hence,

To reduce an improper fraction to a whole or mixed number,

Rule.—*Divide the numerator by the denominator; if there is any remainder, place it over the divisor, and annex the fraction so formed to the quotient.*

Ex. 1. Reduce $\frac{96}{19}$ to a whole or mixed number.

$\frac{96}{19} = 96 \div 19 = 5\frac{1}{19}$, Ans.

2. Reduce $\frac{877}{7}$ to a whole or mixed number. Ans. $125\frac{2}{7}$.
3. Reduce $\frac{744}{8}$. Ans. 93.
4. Reduce $\frac{191}{3}$. Ans. $63\frac{2}{3}$.
5. Reduce $\frac{26}{4}$. Ans. $6\frac{2}{4} = 6\frac{1}{2}$.

(a) The fraction obtained by the above rule will not be in its lowest terms unless the improper fraction is in its lowest terms; for the common measure of the numerator and denominator will also be a common measure of the denominator and remainder (113).

6. Reduce $\frac{197455}{20}$. Ans. $9872\frac{15}{20} = 9872\frac{3}{4}$
7. Reduce $\frac{496870}{25}$.
8. Reduce $\frac{1728}{12}$.
9. Reduce $\frac{7319}{13}$.

10. Reduce $\frac{764}{12}$.
11. Reduce $\frac{76987}{15}$.
12. Reduce $\frac{869432}{1728}$.
13. Reduce $\frac{56987}{176}$.
14. Reduce $\frac{874326}{4327}$.

CASE 7.

131. In $3\frac{1}{4}$ how many 4ths?

Since $\frac{4}{4}$ make a unit, there will be 4 times as many 4ths as units; ∴, in 3 units there will be 4 times 3 fourths $= \frac{12}{4}$, and in $3\frac{1}{4}$ there will be $\frac{12}{4} + \frac{1}{4} = \frac{13}{4}$. Hence,

To reduce a mixed number to an improper fraction,

RULE.—*Multiply the whole number by the denominator of the fraction; to the product add the numerator, and under the sum write the denominator.*

Ex. 1. Reduce $5\frac{4}{7}$ to 7ths. Ans. $\frac{39}{7}$.
2. Reduce $12\frac{7}{11}$ to 11ths. Ans. $\frac{139}{11}$.
3. Reduce $50\frac{3}{4}$ to an improper fraction. Ans. $\frac{203}{4}$.
4. Reduce $130\frac{1}{6}$ to an improper fraction.
5. Reduce $73\frac{5}{9}$. Ans. $\frac{662}{9}$.
6. Reduce $9872\frac{3}{4}$. Ans. $\frac{39491}{4}$.
7. Reduce $19874\frac{4}{5}$.
8. Reduce $7698\frac{43}{47}$.
9. Reduce $6942\frac{87}{157}$.
10. Reduce $46358\frac{57}{895}$.
11. Reduce $276\frac{1}{1728}$.
12. Reduce $3562\frac{2}{1973}$.
13. Reduce $12345\frac{1}{12345}$.
14. Reduce $6789\frac{1}{6789}$.

(a) To reduce an integer to a fraction having any given denominator:—*Multiply the integer by the proposed denominator, and under the product write the denominator* (60).

15. Reduce 9 to a fraction whose denominator is 5.

Ans. $\frac{45}{5}$.

16. Reduce 7 to a fraction whose denominator is 1.

Ans. $\frac{7}{1}$ (127, d).

17. Reduce 87 to a fraction whose denominator is 87.
18. Reduce 7345 to a fraction whose denominator is 372.
19. Reduce 47 to a fraction having 18 for a denominator.
20. Reduce 734 to a fraction having 173 for a denominator.

Case 8.

132. The complex fraction $\frac{\frac{3}{4}}{\frac{5}{7}}$ equals what simple fraction?

The operation required is only to divide a fraction by a fraction; thus, $\frac{\frac{3}{4}}{\frac{5}{7}} = \frac{3}{4} \div \frac{5}{7} = \frac{3}{4} \times \frac{7}{5} = \frac{21}{20}$. Hence,

To reduce a complex fraction to a simple one,

RULE.—*First, if necessary, reduce the numerator and denominator of the complex fraction each to a simple fraction; then divide the fractional numerator by the fractional denominator* (128).

Ex. 1. Reduce $\frac{2\frac{3}{5}}{5\frac{4}{7}}$ to a simple fraction.

$$\frac{2\frac{3}{5}}{5\frac{4}{7}} = \frac{\frac{13}{5}}{\frac{39}{7}} = \frac{13}{5} \div \frac{39}{7} = \frac{13}{5} \times \frac{7}{39} = \frac{7}{15}, \text{ Ans.}$$

2. Reduce $\frac{5\frac{7}{12}}{3\frac{5}{11}}$ to a simple fraction. Ans. $\frac{737}{456}$.

3. Reduce $\frac{8\frac{4}{9}}{15\frac{3}{8}}$ to a simple fraction.

4. Reduce $\frac{\frac{5}{17}}{8\frac{3}{13}}$ to a simple fraction.

5. Reduce $\frac{4\frac{3}{8}}{\frac{2}{8}}$ to a simple fraction.

6. Reduce $\frac{\frac{1}{2} \text{ of } \frac{5}{7} \text{ of } 7\frac{3}{8}}{19\frac{6}{25}}$ to a simple fraction

7. Reduce $\dfrac{\frac{5}{7}}{6}$ to a simple fraction.

(a) The above rule is always applicable, but examples like the 7th may also be reduced by Art. 126; thus,

$$\frac{\frac{5}{7}}{6} = \tfrac{5}{7} \div 6 = \tfrac{5}{42}, \text{ Ans.}$$

This example may also be reduced by multiplying both numerator and denominator of the complex fraction by 7 (60, Cor.); thus,

$$\frac{\frac{5}{7}}{6} = \frac{\frac{5}{7}}{6} \times \tfrac{7}{7} = \tfrac{5}{42}, \text{ Ans. as before.}$$

8. Reduce $\dfrac{\frac{7}{19}}{73}$ to a simple fraction. Ans. $\frac{7}{1387}$.

9. Reduce $\dfrac{\frac{15}{29}}{82}$ to a simple fraction.

10. Reduce $\dfrac{7}{\frac{3}{5}}$ to a simple fraction.

(b) $\dfrac{7}{\frac{3}{5}} = \dfrac{7}{\frac{3}{5}} \times \frac{5}{5} = \frac{35}{3}$; i. e. *a whole number is divided by a fraction* by multiplying the whole number by the denominator, and then dividing the product by the numerator.

11. Reduce $\dfrac{19}{\frac{13}{15}}$ to a simple fraction.

12. Reduce $\dfrac{876}{2\frac{1}{4}}$ to a mixed number.

13. Reduce $\dfrac{18\frac{2}{7}}{\frac{2}{3} \text{ of } \frac{3}{5} \text{ of } \frac{5}{2}}$ to a mixed number.

14. Reduce $\dfrac{\frac{2}{7} \text{ of } \frac{5}{3} \text{ of } \frac{3}{2} \text{ of } \frac{14}{5}}{\frac{3}{11} \text{ of } \frac{22}{4} \text{ of } \frac{8}{6} \text{ of } \frac{1}{2} \text{ of } 2}$ to its simplest form.

Case 9.

133. Fractions having *like* denominators are said to have a *common denominator;* thus, $\frac{3}{8}$ and $\frac{5}{8}$ have a common denominator; so, also, have $\frac{2}{11}$, $\frac{4}{11}$ and $\frac{7}{11}$; but $\frac{2}{3}$ and $\frac{5}{7}$ have not; however, $\frac{2}{3}$ and $\frac{5}{7}$ will be changed to equivalent fractions having a common denominator, if the terms of the 1st be multiplied by 7, and the terms of the 2d by 3; i. e. if the terms of each fraction be multiplied by the denominator of the other (60, Cor.); thus, $\frac{2}{3} = \frac{2}{3} \times \frac{7}{7} = \frac{14}{21}$, and $\frac{5}{7} = \frac{5}{7} \times \frac{3}{3} = \frac{15}{21}$; i. e. $\frac{2}{3}$ and $\frac{5}{7} = \frac{14}{21}$ and $\frac{15}{21}$, fractions that have a common denominator.

Similar explanations may be given when there are more than 2 fractions. Hence,

To reduce fractions to a common denominator,

Rule 1.—*Multiply all the denominators together for a common denominator, and multiply each numerator into the continued product of all the denominators except its own, for new numerators.*

Ex. 1. Reduce $\frac{3}{5}$, $\frac{4}{7}$, $\frac{2}{9}$ and $\frac{3}{8}$ to a common denominator

$5 \times 7 \times 9 \times 8 = 2520$, common denominator,
$3 \times 7 \times 9 \times 8 = 1512$, 1st numerator,
$5 \times 4 \times 9 \times 8 = 1440$, 2d numerator,
$5 \times 7 \times 2 \times 8 = 560$, 3d numerator,
$5 \times 7 \times 9 \times 3 = 945$, 4th numerator;

$\therefore \frac{3}{5}, \frac{4}{7}, \frac{2}{9}$ and $\frac{3}{8} = \frac{1512}{2520}, \frac{1440}{2520}, \frac{560}{2520}$ and $\frac{945}{2520}$, Ans.

2. Reduce $\frac{5}{12}$, $\frac{7}{19}$, $\frac{13}{5}$ and $\frac{11}{17}$ to a common denominator.
Ans. $\frac{8075}{19380}$, $\frac{7140}{19380}$, $\frac{50388}{19380}$ and $\frac{12540}{19380}$.
3. Reduce $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$ and $\frac{5}{7}$. Ans. $\frac{280}{420}$, $\frac{315}{420}$, $\frac{336}{420}$ and $\frac{300}{420}$.
4. Reduce $\frac{4}{7}$, $\frac{5}{11}$, $\frac{2}{5}$ and $\frac{3}{8}$.
5. Reduce $\frac{5}{6}$, $\frac{3}{13}$ and $\frac{7}{17}$.
6. Reduce $\frac{6}{11}$, $\frac{4}{15}$, $\frac{3}{19}$ and $\frac{5}{28}$.
7. Reduce $\frac{2}{5}$, $\frac{2}{7}$, $\frac{2}{9}$, $\frac{2}{11}$ and $\frac{2}{1}$.
8. Reduce $\frac{3}{8}$. $\frac{5}{9}$ and $\frac{7}{23}$.

9. Reduce $\frac{2}{17}$, $\frac{4}{3}$, $\frac{1}{4}$ and $\frac{1}{5}$.
10. Reduce $\frac{3}{7}$, $\frac{3}{5}$, $\frac{5}{8}$ and $\frac{2}{3}$.
11. Reduce $\frac{11}{2}$, $\frac{3}{7}$, $\frac{4}{9}$ and $\frac{1}{5}$.
12. Reduce $\frac{13}{8}$, $\frac{3}{11}$ and $\frac{13}{19}$.
13. Reduce $\frac{5}{17}$, $\frac{3}{19}$ and $\frac{7}{43}$.
14. Reduce $\frac{13}{23}$, $\frac{18}{25}$, $\frac{14}{27}$, $\frac{15}{31}$ and $\frac{2}{11}$.

(a) The above rule is always applicable, but it will not always give the *least* common denominator; this, however, may be effected by the following: —

RULE 2. — *Reduce each fraction, if necessary, to its lowest terms* (129); *find the least common multiple of the denominators* (114) *for a common denominator; and, having divided this multiple by each denominator, multiply the several quotients by the respective numerators, for new numerators.*

NOTE 1. — Each of these rules is founded on the principle that multiplying both terms of a fraction by the same number does not alter its value (60, Cor).

15. Reduce $\frac{3}{8}$, $\frac{5}{12}$, $\frac{7}{18}$ and $\frac{11}{24}$ to their least common denominator.

$$\begin{array}{r|rrrr} & \frac{3}{8}, & \frac{5}{12}, & \frac{7}{18}, & \frac{11}{24} \\ 2) & 8, & 12, & 18, & 24 \\ 2) & 4, & 6, & 9, & 12 \\ 2) & 2, & 3, & 9, & 6 \\ 3) & 1, & 3, & 9, & 3 \\ & & 1, & 3, & 1 \end{array}$$

$2 \times 2 \times 2 \times 3 \times 3 = 72$, least common multiple of denominators.

$\frac{72}{8} \times 3 = 27$, 1st numerator.
$\frac{72}{12} \times 5 = 30$, 2d numerator.
$\frac{72}{18} \times 7 = 28$, 3d numerator.
$\frac{72}{24} \times 11 = 33$, 4th numerator.

$\therefore \frac{3}{8}, \frac{5}{12}, \frac{7}{18}$ and $\frac{11}{24} = \frac{27}{72}, \frac{30}{72}, \frac{28}{72}$ and $\frac{33}{72}$, Ans.

16. Reduce $\frac{7}{15}$, $\frac{11}{18}$, $\frac{6}{25}$ and $\frac{3}{35}$.
Ans. $\frac{1470}{3150}$, $\frac{1925}{3150}$, $\frac{756}{3150}$ and $\frac{270}{3150}$.
17. Reduce $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{12}$ and $\frac{3}{8}$.
18. Reduce $\frac{3}{7}$, $\frac{5}{14}$, $\frac{9}{28}$ and $\frac{4}{35}$.
19. Reduce $\frac{3}{22}$, $\frac{5}{11}$, $\frac{8}{77}$ and $\frac{7}{66}$.
20. Reduce $\frac{8}{15}$, $\frac{3}{20}$, $\frac{7}{10}$ and $\frac{11}{45}$.

21. Reduce $\frac{3}{17}$, $\frac{5}{51}$, $\frac{9}{34}$ and $\frac{5}{34}$.
22. Reduce $\frac{7}{9}$, $\frac{13}{15}$, $\frac{8}{27}$ and $\frac{5}{36}$.
23. Reduce $\frac{2}{3}$, $\frac{4}{9}$, $\frac{5}{6}$, $\frac{2}{15}$ and $\frac{7}{27}$.
24. Reduce $\frac{8}{23}$, $\frac{5}{18}$ and $\frac{17}{46}$.

NOTE 2. — The first clause of Rule 2 is omitted by many authors, but its necessity is apparent from

Ex. 25. Reduce $\frac{6}{8}$, $\frac{4}{12}$ and $\frac{3}{18}$ to equivalent fractions having the least common denominator.

Disregarding the first clause of the rule, we find 72 to be the least common multiple of the denominators, and the fractions, $\frac{6}{8}$, $\frac{4}{12}$ and $\frac{3}{18}$, reduce to $\frac{54}{72}$, $\frac{24}{72}$ and $\frac{12}{72}$; but, regarding the first clause, we have $\frac{6}{8}$, $\frac{4}{12}$ and $\frac{3}{18} = \frac{3}{4}$, $\frac{1}{3}$ and $\frac{1}{6} = \frac{9}{12}$, $\frac{4}{12}$ and $\frac{2}{12}$, which have a common denominator less than 72.

26. Reduce $\frac{2}{4}$, $\frac{6}{16}$ and $\frac{3}{24}$. Ans. $\frac{4}{8}$, $\frac{3}{8}$ and $\frac{1}{8}$.
27. Reduce $\frac{2}{6}$, $\frac{5}{20}$ and $\frac{10}{15}$.
28. Reduce $\frac{8}{16}$, $\frac{3}{18}$ and $\frac{6}{24}$.
29. Reduce $\frac{3}{21}$ and $\frac{7}{28}$.
30. Reduce $\frac{3}{36}$ and $\frac{8}{72}$.

NOTE 3. — In this and the following cases, each fraction should, if necessary, be reduced to a simple form before applying the rule.

31. Reduce $\frac{2}{3}$ of $\frac{7}{8}$ and $\dfrac{3\frac{4}{7}}{5\frac{1}{6}}$ to a common denominator.

$\frac{2}{3}$ of $\frac{7}{8} = \frac{7}{12}$; $\dfrac{3\frac{4}{7}}{5\frac{1}{6}} = \frac{25}{7} \div \frac{31}{6} = \frac{150}{217}$; but

$\frac{7}{12}$ and $\frac{150}{217} = \frac{1519}{2604}$ and $\frac{1800}{2604}$, Ans.

32. Reduce $\dfrac{5\frac{2}{3}}{7}$ and $5\frac{3}{5}$ to fractions having a common denominator. Ans. $\frac{85}{105}$ and $\frac{588}{105}$.

33. Reduce $\dfrac{\frac{8}{9} \text{ of } 4\frac{3}{7}}{63\frac{2}{17}}$, $\dfrac{5\frac{3}{11}}{\frac{1}{2} \text{ of } \frac{3}{12}}$ and $\dfrac{\frac{1}{3} \text{ of } \frac{5}{8} \text{ of } 7}{3 \times \frac{1}{7} \times 4\frac{2}{11}}$.

34. Reduce $\frac{1}{3}$ of $\frac{8}{7}$, $2\frac{1}{3}$, $\dfrac{3\frac{1}{4}}{5\frac{7}{12}}$ and $\frac{4}{7}$.

35. Reduce 5, $6\frac{2}{3}$, $\frac{8}{9}$ of $\frac{2}{3}$, $\dfrac{\frac{1}{5}}{\frac{7}{8}}$ and $\frac{3}{4}$.

CASE 10.

134. As 1 penny is equal to 4 farthings, so any fraction of a penny is 4 times as great a fraction of a farthing; e. g. $\frac{1}{2}$d. $=$ $\frac{4}{2}$qr., $\frac{1}{5}$d. $=$ $\frac{4}{5}$qr., etc. Hence,

To reduce a fraction of a higher denomination to one of a lower,

RULE. — *Multiply the fraction by such numbers as are necessary to reduce the given to the required denomination.*

Ex. 1. Reduce $\frac{7}{36}$s. to the fraction of a farthing.

$\frac{7}{36}$s. $(= \frac{7}{36}\text{d.} \times 12) = \frac{7}{3}$d. $(= \frac{7}{3}\text{qr.} \times 4) = \frac{28}{3}$qr., Ans.; or,

$$\frac{7 \times 12 \times 4}{36} = \frac{7 \times \cancel{12} \times 4}{\cancel{36} \quad 3} = \frac{28}{3}\text{qr., Ans. as before.}$$

2. Reduce $\frac{7}{240}$ of a ton to the fraction of a dram.

$$\frac{7 \times 20 \times 4 \times 25 \times 16 \times 16}{240} = \frac{7 \times \cancel{20} \times \cancel{4} \times 25 \times 16 \times 16}{\cancel{240} \quad \cancel{12} \quad 3}$$

$$= \frac{44800}{3}\text{dr., Ans.}$$

3. Reduce $\frac{10}{21}$ of a rod to the fraction of a barleycorn.

$$\frac{10 \times 16\frac{1}{2} \times 12 \times 3}{21} = \frac{10 \times \overset{11}{\cancel{33}} \times \overset{6}{\cancel{12}} \times 3}{\underset{7}{\cancel{21}} \times \cancel{2}} = \frac{1980}{7}\text{ b. c., Ans.}$$

4. Reduce $\frac{7}{600}$ of a pound Troy to the fraction of a grain. Ans. $\frac{336}{5}$.

5. Reduce $\frac{7}{600}$ of a pound Apothecaries' weight to the fraction of a grain. Ans. $\frac{336}{5}$.

6. Reduce $\frac{1}{560}$ of a day to the fraction of a second.

7. Reduce $\frac{1}{6}$ of a bush. to the fraction of a pint.

8. Reduce $\frac{1}{23}$gal. to the fraction of a gill.

9. Reduce $\frac{2}{21}$yd. to the fraction of an inch, cubic measure.

10. Reduce $\frac{1}{525}$ sign to the fraction of a second.

11. Reduce $\frac{3}{73}$sq. m. to the fraction of an inch.

12. Reduce $\frac{10}{31}$m. to the fraction of a link.
13. Reduce $\frac{1}{1814400}$wk. to the fraction of a second.
14. Reduce $\frac{4}{225}$acres to the fraction of a square yard.
15. Reduce $\frac{5}{225}$yd. of cloth to the fraction of an inch.
16. Reduce $\frac{1}{21}$circ. to the fraction of a second.

Case 11.

135. In 15 barleycorns there is only $\frac{1}{3}$ of 15 inches, so in $\frac{2}{5}$ of a barleycorn there is only $\frac{1}{3}$ of $\frac{2}{5}$ of an inch; in $\frac{3}{7}$ of a peck there is but $\frac{1}{4}$ of $\frac{3}{7}$ of a bushel, etc. Hence,

To reduce a fraction of a lower to a fraction of a higher denomination,

Rule.—*Divide the given fraction by such numbers as are required to reduce the given to the required denomination.*

Ex. 1. Reduce $\frac{28}{3}$qr. to the fraction of a shilling.

$\frac{28}{3}$qr. $(= \frac{28}{3}\text{d.} \div 4) = \frac{7}{3}$d. $(= \frac{7}{3}\text{s.} \div 12) = \frac{7}{36}$s., Ans.; or,

$$\frac{\cancel{28}\ \ 7}{3 \times \cancel{4} \times 12} = \frac{7}{36}\text{s., Ans. as before.}$$

2. Reduce $\frac{44800}{3}$dr. to the fraction of a ton.

$$\frac{\cancel{44800}\ \ \cancel{2800}\ \ \cancel{175}\ \ 7}{3 \times \cancel{16} \times \cancel{16} \times \cancel{25} \times 4 \times 20} = \frac{7}{240}\text{ tons, Ans}$$

3. Reduce $\frac{1980}{7}$b. c. to the fraction of a rod.

$$\frac{1980}{7 \times 3 \times 12 \times 3 \times 5\frac{1}{2}} = \frac{\cancel{1980}\ \ \cancel{180}\ \ \overset{10}{\cancel{30}}\ \ \cancel{2}}{7 \times 3 \times \underset{6}{\cancel{12}} \times \cancel{3} \times \cancel{11}} = \frac{10}{21}\text{rd., Ans.}$$

4. Reduce $\frac{336}{5}$gr. to the fraction of a pound Troy weight.
Ans. $\frac{7}{600}$.

5. Reduce $\frac{336}{5}$gr. to the fraction of a pound Apothecaries weight.

6. Reduce $\frac{1080}{7}$sec. to the fraction of a day.

7. Reduce $\frac{32}{3}$pt. to the fraction of a bushel.

8 Reduce $\frac{32}{23}$ gills to the fraction of a gallon.

9. Reduce $\frac{31104}{7}$c. in. to the fraction of a yard.

10. Reduce $\frac{1440}{7}$sec. to the fraction of a sign.

11. Reduce $\frac{1204346880}{73}$sq. in. to the fraction of a mile.

12. Reduce $\frac{152000}{31}$ links to the fraction of a mile.

13. Reduce $\frac{1}{3}$sec. to the fraction of a week.

14. Reduce $\frac{3872}{45}$yds. to the fraction of an acre.

15. Reduce $\frac{4}{5}$in. to the fraction of a yard.

16. Reduce $\frac{432000}{7}$sec. to the fraction of a circumference.

Case 12.

136. $\frac{1}{6}$£ $= \frac{20}{6}$s. $= \frac{10}{3}$s. $= 3\frac{1}{3}$s.; again, $\frac{1}{3}$s. $= \frac{12}{3}$d. $= 4$d.; $\therefore \frac{1}{6}$£ $= 3$s. $+ 4$d. Hence,

To reduce a fraction of a higher denomination to whole numbers of lower denominations,

Rule.—*Reduce the given fraction to a fraction of the next lower denomination by Case* 10; *then, if the fraction is improper, reduce it to a whole or mixed number by Case* 6. *If the result is a mixed number, reduce the fractional part of it to the next lower denomination, as before, and so proceed as far as desirable.*

Note.—If, at any time, the reduced fraction is *proper*, there will be **no** whole number of that denomination.

Ex. 1. Reduce $\frac{11}{24}$ of a shilling to pence and farthings?

$\frac{11}{24}$s. ($= \frac{11}{24}$d. $\times 12) = \frac{11}{2}$d. $= 5\frac{1}{2}$d., and $\frac{1}{2}$d. ($= \frac{1}{2}$qr. $\times 4) = 2$ qr.

$\therefore \frac{11}{24}$s. $= 5$d. $+ 2$qr., Ans.

2. Reduce $\frac{5}{22}$ of a furlong to rods, yards, etc.

$\frac{5}{22}$ fur. ($= \frac{5}{22}$rd. $\times 40) = \frac{100}{11}$rd. $= 9\frac{1}{11}$rd.; again,

$\frac{1}{11}$rd. ($= \frac{1}{11}$yd. $\times 5\frac{1}{2} = \frac{1}{11} \times \frac{11}{2}) = \frac{1}{2}$yd., a proper fraction;

again, $\frac{1}{2}$yd. ($= \frac{1}{2}$ft. $\times 3) = \frac{3}{2}$ft. $= 1\frac{1}{2}$ft.; and, finally,

$\frac{1}{2}$ft. ($= \frac{1}{2}$in. $\times 12) = 6$ in.;

$\therefore \frac{5}{22}$fur. $= 9$rd. 0yd. 1ft. 6in., Ans.

3. What is the value of $\frac{8}{15}$ of an acre;

Ans. 2r. 5rd. 10yd. 0ft. 108in.

4. What is the value of $\frac{9}{25}$ of a pound Troy?
5. What is the value of $\frac{7}{96}$ of a pound, Apothecaries' weight?
6. Reduce $\frac{73}{512}$lb., Avoirdupois, to ounces and drams.
7. Reduce $\frac{3}{17}$ of a mile to furlongs, chains, etc.
8. Reduce $\frac{13}{20}$ of a cord to cord feet, cubic feet, etc.
9. Reduce $\frac{6}{19}$ of a yard to quarters, nails, etc.
10. Reduce $\frac{53}{97}$ of a gallon to quarts, pints, etc.
11. Reduce $\frac{21}{38}$bush. to pecks, quarts, etc.
12. Reduce $\frac{1}{7}$ of a Julian year to lunar months, etc.
13. Reduce $\frac{1}{9}$circ. to signs, degrees, etc.
14. Reduce $\frac{3}{7}$ of a league to miles, furlongs, etc.
15. Reduce $\frac{5}{24}$ of a ton to cubic feet and inches.
16. Reduce $\frac{19}{27}$ of a civil year (365 days) to days, etc.
17. Reduce $\frac{2879}{2880}$lb. to ounces, drams, scruples, etc.
18. Reduce $\frac{47791}{64800}$circ. to signs, degrees, etc.

Case 13.

137. 3 farthings are the same as $\frac{3}{4}$ of a penny; again, 6d. $+$ 1qr. $=$ 25qr., and 1s. $=$ 48 qr.; $\therefore$ 6d. and 1qr. $= \frac{25}{48}$s. Hence,

To reduce whole numbers of lower denominations to the fraction of a higher denomination,

Rule 1.—*Reduce the given quantity to the lowest denomination it contains, for a numerator; and reduce a unit of the higher denomination to the same denomination as the numerator, for a denominator.*

Ex. 1. Reduce 5d. and 2qr. to the fraction of a shilling.

$$5\text{d.} + 2\text{qr.} = 22\text{qr., and } 1\text{s.} = 48\text{qr.};$$

$$\therefore 5\text{d.} + 2\text{qr.} = \tfrac{22}{48}\text{s.} = \tfrac{11}{24}\text{s., Ans.}$$

2. Reduce 9 rods, 1 foot and 6 inches to the fraction of a furlong.

$$9\text{rd. } 1\text{ft. } 6\text{in.} = 1800\text{in. and } 1\text{fur.} = 7920\text{in.}; \therefore$$

$$9\text{rd. } 1\text{ft. } 6\text{in.} = \tfrac{1800}{7920}\text{fur.} = \tfrac{5}{22}\text{fur., Ans.}$$

(a) In Ex. 1, 2qr. are equal to $\frac{1}{2}$d.; $\therefore$ 5d. and 2qr. are equal to $5\frac{1}{2}$d. $= \frac{11}{2}$d. $= \frac{1}{12}$ of $\frac{11}{2}$s. $= \frac{11}{24}$s., Ans.

In Ex. 2, 6in. $= \frac{1}{2}$ft.; $1\frac{1}{2}$ft. $= \frac{1}{2}$yd. $= \frac{1}{11}$rd. and $9\frac{1}{11}$rd. $= \frac{100}{11}$rd. $= \frac{5}{22}$fur., Ans. as by Rule 1. Hence,

RULE 2.—*Divide the number of the lowest denomination given by the number required to reduce it to the next higher denomination, and annex the fractional quotient so obtained to the given number of that higher denomination; divide the mixed number so formed by the number required to reduce it to the* NEXT *higher denomination, annex the quotient to the given number of that denominaton, and so proceed as far as necessary.*

NOTE 1.—This rule is frequently preferable to the 1st, because it enables us to use smaller numbers and gives the result in lower terms.

3. Reduce 2r. 5rd. 10yd. 0ft. 108in. to the fraction of an acre.
Ans. $\frac{8}{15}$.

4. Reduce 4oz. 6dwt. $9\frac{3}{5}$gr. to the fraction of a pound.
Ans. $\frac{9}{25}$.

NOTE 2.—In Example 4, by Rule 1, reduce 4oz. 6dwt. $9\frac{3}{5}$ gr. to *fifths* of a grain for a numerator, and 1lb. to fifths of a grain for a denominator How shall it be done by Rule 2? Which mode is preferable? Why?

5. Reduce 73 to the fraction of a pound.

6. Reduce 2oz. $4\frac{1}{2}$dr. to the fraction of a pound.

7. Reduce 1fur. 4ch. 11li. $6\frac{24}{425}$in. to the fraction of a mile.

8. Reduce 7 cord feet, 9 cubic feet and $1036\frac{4}{5}$ inches to the fraction of a cord.

9. Reduce 1qr. 1na. $\frac{9}{76}$in. to the fraction of a yard.

10. Reduce 2qt. $1\frac{47}{97}$gi. to the fraction of a gallon.

11. Reduce 2pk. 1qt. $1\frac{7}{19}$pt. to the fraction of a bushel.

12. Reduce 1 l.m. 3wk. 3d. 4h. 17m. $8\frac{4}{7}$sec. to the fraction of a Julian year.

13. Reduce 1s. 10° to the fraction of a circumference.

14. Reduce 1m. 2fur. 11 rd. 2yd. 1ft. $2\frac{4}{7}$b. c. to the fraction of a league.

15. Reduce 8ft. 576in. to the fraction of a ton.

16. Reduce 256d. 20h. 26m. 40sec. to the fraction of a civil year (365 days).

17. Reduce 11℥ 7ʒ 2℈ 18gr. to the fraction of a pound.

18. Reduce 8s. 25° 30′ 20″ to the fraction of a circumference

CASE 14.

138. If numbers of the same kind are added together, their sum will be of the same kind as the numbers added; thus, 3 books $+$ 4 books $=$ 7 books; 3 hats $+$ 4 hats $=$ 7 hats; and for a like reason, $\frac{3}{5}+\frac{4}{5}=\frac{7}{5}$; $\frac{5}{13}+\frac{4}{13}=\frac{9}{13}$, etc., etc.

(a) Numbers of different kinds cannot be united by addition; thus, 3 hats $+$ 4 books are neither 7 hats nor 7 books; so $\frac{3}{5}+\frac{4}{9}$ are neither $\frac{7}{5}$ nor $\frac{7}{9}$; but numbers that are unlike may sometimes be made alike by reduction, and then added; thus,

$$\frac{3}{5}+\frac{4}{9}=\frac{27}{45}+\frac{20}{45}\ (133)=\frac{47}{45}.$$

(b) Again, 2bush. $+$ 3pk. are neither 5bush. nor 5pk.; but 2bush. $=$ 8pk., and then 8pk. $+$ 3pk. $=$ 11pk.; so $\frac{2}{7}$bush. $+$ $\frac{3}{7}$pk. are neither $\frac{5}{7}$bush. nor $\frac{5}{7}$pk.; but $\frac{2}{7}$bush. $=\frac{8}{7}$pk. (134), and then $\frac{8}{7}$pk. $+\frac{3}{7}$pk. $=\frac{11}{7}$pk. Hence,

To add fractions,

RULE.—*Reduce the fractions, if necessary, first to the same denomination, then to a common denominator; after which, write the sum of the numerators over the common denominator.*

Ex. 1. Add $\frac{5}{17}$ and $\frac{6}{17}$ together. Ans. $\frac{11}{17}$.

2. Add $\frac{3}{19}$, $\frac{7}{19}$ and $\frac{4}{19}$ together. Ans. $\frac{14}{19}$.

3. Add $\frac{8}{13}$, $\frac{6}{13}$, $\frac{7}{13}$ and $\frac{11}{13}$ together. Ans. $\frac{32}{13}=2\frac{6}{13}$.

4. Add $\frac{7}{12}$ and $\frac{11}{12}$ together. Ans. $\frac{18}{12}=1\frac{6}{12}=1\frac{1}{2}$.

5. Add $\frac{7}{15}$, $\frac{8}{15}$, $\frac{2}{15}$ and $\frac{4}{15}$ together. Ans. $1\frac{2}{5}$.

6. Add $\frac{3}{20}$, $\frac{13}{20}$, $\frac{7}{20}$ and $\frac{9}{20}$ together.

7. Add $\frac{73}{200}$, $\frac{83}{200}$, $\frac{69}{200}$ and $\frac{25}{200}$ together.

8. Add $\frac{5}{28}$, $\frac{6}{28}$, $\frac{7}{28}$, $\frac{9}{28}$, $\frac{11}{28}$ and $\frac{27}{28}$ together.

9. Add $\frac{7}{25}$, $\frac{9}{25}$, $\frac{3}{25}$, $\frac{12}{25}$ and $\frac{6}{25}$ together.

10. Add $\frac{3}{72}$, $\frac{19}{72}$, $\frac{39}{72}$, $\frac{78}{72}$, $\frac{94}{72}$ and $\frac{33}{72}$ together.

11. Add together $\frac{3}{8}$, $\frac{7}{12}$ and $\frac{5}{16}$.

$\frac{3}{8}+\frac{7}{12}+\frac{5}{16}=\frac{18}{48}+\frac{28}{48}+\frac{15}{48}$ (133, a) $=\frac{61}{48}=1\frac{13}{48}$, Ans.

12. Add together $\frac{3}{8}$, $\frac{4}{9}$, $\frac{7}{11}$ and $\frac{2}{5}$.

$\frac{3}{8}+\frac{4}{9}+\frac{7}{11}+\frac{2}{5}=\frac{1485}{3960}+\frac{1760}{3960}+\frac{2520}{3960}+\frac{1584}{3960}$ (133) $=$ $1\frac{3389}{3960}$, Ans.

13. Add together $\frac{5}{7}$, $\frac{3}{5}$, $\frac{2}{3}$ and $\frac{7}{19}$. Ans. $2\frac{697}{1995}$.

14. Add together $\frac{3}{11}$, $\frac{7}{22}$, $\frac{5}{9}$ and $\frac{3}{20}$.

15. What is the sum of $\frac{5}{6}$, $\frac{8}{12}$, $\frac{7}{9}$ and $\frac{15}{26}$?

16. What is the sum of $\frac{3}{8}$, $\frac{5}{12}$, $\frac{7}{11}$, $\frac{5}{9}$ and $\frac{4}{3}$?

17. What is the sum of $\frac{25}{54}$, $\frac{8}{11}$, $\frac{7}{33}$ and $\frac{39}{44}$?

18. What is the sum of $\frac{3}{4}$ of $\frac{4}{5}$, $\frac{7}{8}$ of $\frac{8}{9}$ of $\frac{9}{10}$ and $\frac{1}{2}$ of $\frac{4}{5}$? Ans. $1\frac{7}{10}$.

19. Add together $\dfrac{3\frac{1}{4}}{5\frac{2}{3}}$, $\frac{7}{8}$ and $\frac{9}{10}$ of $\frac{6}{7}$. Ans. $2\frac{1047}{4760}$.

20. Add together $\dfrac{5}{3\frac{1}{3}}$, $\frac{7}{8}$ of $\frac{4}{5}$, $\frac{9}{5}$ and $\frac{4}{7}$.

21. Add together $\frac{7}{8}$ of $\frac{3}{9}$, $\dfrac{7\frac{3}{5}}{2\frac{4}{5}}$, $\dfrac{8\frac{3}{7}}{9}$ and $\frac{1}{3}$.

22. Add $\frac{1}{5}$s. to $\frac{1}{3}$d.

$\frac{1}{5}$s. $+\frac{1}{3}$d. $=\frac{12}{5}$d. $+\frac{1}{3}$d. $=\frac{36}{15}$d. $+\frac{5}{15}$d. $=\frac{41}{15}$d. $=2\frac{11}{15}$d., Ans.;

or, $\frac{1}{5}$s. $+\frac{1}{3}$d. $=\frac{1}{5}$s. $+\frac{1}{36}$s. $=\frac{36}{180}$s. $+\frac{5}{180}$s. $=\frac{41}{180}$s., 2d Ans. $=$ 1st Ans.

23. Add $\frac{1}{3}$gal., $\frac{1}{5}$qt. and $\frac{2}{7}$pt. together.

Note. — The answer to the 23d example may be the fraction of a gallon, of a quart or of a pint; thus,

1st, $\frac{1}{3}$gal. $+\frac{1}{5}$qt. $+\frac{2}{7}$pt. $=\frac{1}{3}$gal. $+\frac{1}{20}$gal. $+\frac{1}{28}$gal. $=\frac{140}{420}$gal. $+\frac{21}{420}$gal. $+\frac{15}{420}$gal. $=\frac{176}{420}$gal. $=\frac{44}{105}$gal., 1st Ans.

2d, $\frac{1}{3}$gal. $+\frac{1}{5}$qt. $+\frac{2}{7}$pt. $=\frac{4}{3}$qt. $+\frac{1}{5}$qt $+\frac{1}{7}$qt. $=\frac{140}{105}$qt. $+\frac{21}{105}$qt. $+\frac{15}{105}$qt. $=\frac{176}{105}$qt. $=1\frac{71}{105}$qt., 2d Ans.

3d, $\frac{1}{3}$gal. $+\frac{1}{5}$qt. $+\frac{2}{7}$pt. $=\frac{8}{3}$pt. $+\frac{2}{5}$pt. $+\frac{2}{7}$pt. $=\frac{280}{105}$pt. $+\frac{42}{105}$pt. $+\frac{30}{105}$pt. $=\frac{352}{105}=3\frac{37}{105}$pt., 3d Ans.

24. Add together $\frac{1}{7}$ of a ton and $\frac{1}{5}$ of a hundred weight.

25. Add together $\frac{2}{3}$bush. $\frac{3}{5}$pk. and $\frac{4}{7}$qt.

26. $\frac{1}{5}$wk. $+$ $\frac{2}{5}$d. $+$ $\frac{1}{7}$h. $+$ $\frac{3}{7}$m. $=$ what fraction of a week? day? hour? minute? 3d Ans. $\frac{8667}{20}$h. $= 43\frac{7}{20}$h.

27. Add together $\frac{2}{5}$lb. $\frac{1}{7}$℥ $\frac{2}{7}$ʒ and $\frac{3}{8}$℈.

28. Add together $\frac{1}{7}$ and $\frac{1}{9}$.

$$\frac{1}{7} + \frac{1}{9} = \frac{9+7}{7 \times 9} = \frac{16}{63}, \text{ Ans.}$$

29. Add together $\frac{2}{7}$ and $\frac{2}{9}$.

$$\frac{2}{7} + \frac{2}{9} = \frac{(9+7) \times 2}{7 \times 9} = \frac{32}{63}, \text{ Ans.}$$

(c) What is the easiest mode of adding *two* fractions that have a *common numerator?*

30. Add together $\frac{3}{11}$ and $\frac{3}{15}$.
31. Add together $\frac{7}{8}$ and $\frac{7}{11}$.
32. Add together $\frac{11}{5}$ and $\frac{11}{6}$.
33. Add together $\frac{13}{4}$ and $\frac{13}{9}$.

(d) Mixed numbers may be added by first reducing them to improper fractions and then to a common denominator, etc.; but it will be easier and ∴ better to add the fractional parts and then add that sum to the sum of the integral parts.

34. Add $3\frac{4}{5}$ and $4\frac{7}{8}$. Ans. $8\frac{27}{40}$.
35. Add $5\frac{7}{8}$, $4\frac{9}{11}$ and $16\frac{2}{3}$.
36. Add $8\frac{4}{5}$, $2\frac{7}{9}$, $3\frac{5}{8}$ and $4\frac{1}{5}$.
37. Add $7\frac{4}{9}$, $3\frac{2}{4}$, $5\frac{3}{8}$, $4\frac{2}{3}$ and $5\frac{5}{6}$.
38. Add $\dfrac{4\frac{3}{5}}{2\frac{1}{3}}$, $\dfrac{7\frac{1}{3}}{\frac{1}{5}}$, $4\frac{1}{7}$ and $\frac{1}{3}$ of $\frac{27}{2}$.

CASE 15.

139. To subtract a less fraction from a greater,

RULE.—*Prepare the fractions as in addition, and then write the difference of the numerators over the common denominator.*

Ex. 1. From $\frac{5}{8}$ subtract $\frac{3}{8}$. Ans. $\frac{2}{8} = \frac{1}{4}$.

2. From $\frac{34}{93}$ take $\frac{5}{93}$. Ans. $\frac{29}{93}$.

3. From $\frac{18}{25}$ take $\frac{3}{25}$.
4. From $\frac{75}{97}$ take $\frac{17}{97}$.
5. From $\frac{25}{49}$ take $\frac{18}{49}$.
6. From $\frac{24}{57}$ take $\frac{14}{57}$.
7. From $\frac{27}{27}$ take $\frac{17}{27}$.
8. From $\frac{7}{8}$ take $\frac{2}{5}$.

(a) $\frac{7}{8} - \frac{2}{5} = \frac{35}{40} - \frac{16}{40} = \frac{19}{40}$, Ans. (See 138, a.)

9. From $\frac{19}{20}$ take $\frac{17}{53}$. Ans. $\frac{667}{1060}$.
10. From $\frac{18}{73}$ take $\frac{15}{96}$.
11. From $\frac{27}{15}$ take $\frac{14}{13}$.
12. From $\frac{27}{29}$ take $\frac{1}{2}$.
13. From $\frac{19}{87}$ take $\frac{13}{96}$.
14. From $\frac{27}{85}$ take $\frac{13}{79}$.
15. From $\frac{3}{4}$ of $\frac{4}{5}$ take $\frac{1}{2}$ of $\frac{2}{3}$.

$\frac{3}{4}$ of $\frac{4}{5} - \frac{1}{2}$ of $\frac{2}{3} = \frac{3}{5} - \frac{1}{3} = \frac{9}{15} - \frac{5}{15} = \frac{4}{15}$, Ans.

16. From $\frac{4}{7}$ of $\frac{9}{6}$ of $\frac{8}{12}$ take $\frac{3}{11}$ of $\frac{8}{25}$ of $\frac{75}{24}$. Ans. $\frac{23}{77}$.
17. From $\frac{5}{7}$ of $\frac{8}{15}$ of $\frac{14}{12}$ take $\frac{7}{8}$ of $\frac{12}{21}$ of $\frac{2}{11}$.
18. From $\dfrac{3\frac{1}{5}}{4\frac{2}{7}}$ take $\dfrac{5\frac{1}{6}}{9\frac{2}{5}}$.

$$\left.\begin{array}{l} \dfrac{3\frac{1}{5}}{4\frac{2}{7}} = \dfrac{\frac{16}{5}}{\frac{30}{7}} = \frac{16}{5} \div \frac{30}{7} = \frac{16}{5} \times \frac{7}{30} = \frac{56}{75}; \\ \dfrac{5\frac{1}{6}}{9\frac{2}{5}} = \dfrac{\frac{31}{6}}{\frac{47}{5}} = \frac{31}{6} \div \frac{47}{5} = \frac{31}{6} \times \frac{5}{47} = \frac{155}{282}; \end{array}\right\} \text{Complex fractions reduced to simple ones.}$$

$$\frac{56}{75} - \frac{155}{282} = \frac{5264}{7050} - \frac{3875}{7050} = \frac{1389}{7050} = \frac{463}{2350}, \text{ Ans.}$$

19. From $\dfrac{7\frac{2}{3}}{7\frac{4}{5}}$ take $\dfrac{4\frac{2}{7}}{5\frac{8}{9}}$
20. From $\frac{1}{5}$s. take $\frac{1}{3}$d.

(b) $\frac{1}{5}$s. — $\frac{1}{3}$d. $=\frac{12}{5}$d. — $\frac{1}{3}$d. $= \frac{36}{15}$d. — $\frac{5}{15}$d. $= \frac{31}{15}$d. $= 2\frac{1}{15}$d., Ans.
or, $\frac{1}{5}$s. — $\frac{1}{3}$d. $= \frac{1}{5}$s. — $\frac{1}{36}$s. $= \frac{36}{180}$s. — $\frac{5}{180}$s. $= \frac{31}{180}$s., 2d Ans.
(138, b.)

21. From $\frac{1}{3}$gal. take $\frac{1}{5}$qt. Ans. $\frac{17}{60}$gal. or $1\frac{2}{15}$qt.
22. From $\frac{1}{7}$ton take $\frac{1}{5}$cwt.
23. From $\frac{1}{7}$wk. take $\frac{1}{7}$h.
24. From $\frac{2}{5}$d. take $\frac{3}{7}$m.

25. From $\frac{2}{5}$lb. take $\frac{3}{8}$℈.

26. From $\frac{1}{7}$ take $\frac{1}{9}$.

$$(c)\ \frac{1}{7}-\frac{1}{9}=\frac{9-7}{7\times 9}=\frac{2}{63}, \text{ Ans.} \quad (138, c.)$$

27 From $\frac{2}{7}$ take $\frac{2}{9}$.

$$\frac{2}{7}-\frac{2}{9}=\frac{(9-7)\times 2}{7\times 9}=\frac{4}{63}, \text{ Ans}$$

28. From $\frac{3}{11}$ take $\frac{3}{15}$.

29. From $\frac{7}{8}$ take $\frac{7}{11}$.

30. From $\frac{11}{5}$ take $\frac{11}{6}$.

31. From $\frac{61}{79}$ take $\frac{61}{125}$.

32. From $\frac{42}{73}$ take $\frac{42}{83}$.

33. From $8\frac{4}{11}$ take $2\frac{7}{11}$.

(d) $8\frac{4}{11}-2\frac{7}{11}=7\frac{15}{11}-2\frac{7}{11}=5\frac{8}{11}$, Ans. (138, d.)

34. From $9\frac{1}{3}$ take $1\frac{3}{4}$.

$$9\frac{1}{3}-1\frac{3}{4}=8\frac{4}{3}-1\frac{3}{4}=8\frac{16}{12}-1\frac{9}{12}=7\frac{7}{12}, \text{ Ans.}$$

35. From $19\frac{3}{8}$ take $14\frac{7}{11}$. Ans. $4\frac{65}{88}$.

36. From $46\frac{5}{16}$ take $27\frac{4}{19}$.

37. From $27\frac{4}{5}$ take $13\frac{2}{3}$.

38. From $146\frac{7}{18}$ take $24\frac{9}{16}$.

39. From 276 take $72\frac{3}{5}$.

40. From $82\frac{4}{9}$ take 71.

140. Miscellaneous Examples in Fractions.

Ex. 1. Divide $\frac{3\frac{1}{4}}{6\frac{1}{2}}\times 72\frac{1}{2}$ by $\frac{2}{3}$ of $\frac{3}{5}$ of $9\frac{3}{8}$. Ans. $9\frac{2}{3}$.

2. Multiply $\frac{8\frac{2}{7}}{4\frac{1}{7}}\div\frac{1}{2}$ by $\frac{1}{4}$ of $8\frac{2}{3}$. Ans. $8\frac{2}{3}$.

3. Add $\frac{3}{5}$£ $\frac{2}{7}$s. $\frac{1}{3}$d. and $\frac{2}{5}$qr. together.

4. Reduce $4\frac{1}{5}$ rods, $2\frac{1}{7}$ yards, $2\frac{1}{4}$ feet and $7\frac{1}{3}$ inches to the fraction of a furlong.

5. Subtract $\frac{1}{2}$ of $\frac{9}{11}$ from $\frac{8\frac{2}{3}}{4\frac{1}{5}}\times 2\frac{1}{10}$.

6. Reduce $\frac{2}{7}$ of a gallon to whole numbers of lower denominations.

7. If 1lb. of veal is worth $12\frac{1}{2}$ cents, what is the value of $12\frac{1}{2}$ lb.?

8. A firkin contains $33\frac{1}{2}$lb. of butter; what is the value of the butter at $23\frac{1}{2}$ cents per lb.?

9. If $\frac{7}{8}$ of a bushel of wheat is worth $\$1\frac{3}{4}$, what is the value of $25\frac{2}{5}$ bushels?

10. Paid $\$6\frac{9}{16}$ for $8\frac{3}{4}$ bushels of potatoes; what should I pay for $27\frac{1}{2}$ bushels?

11. Divide $144 among 5 men and 7 boys, so that each man shall receive $\frac{7}{5}$ as much as a boy; what will each receive?

Ans. Each man, $\$14\frac{2}{5}$; each boy, $\$10\frac{2}{7}$.

12. If 15bbl. of flour cost $\$157\frac{1}{2}$, what will $7\frac{1}{2}$bbl. cost?

13. If $2\frac{1}{2}$ cords of wood will pay for $33\frac{1}{3}$ gallons of molasses, what quantity of wood will pay for $7\frac{1}{2}$ times $33\frac{1}{3}$ gallons of molasses? Ans. $18\frac{3}{4}$ cords.

14. If $\frac{6}{11}$ of a bushel of oats cost 42 cents, how many oats may be bought for $\$8\frac{17}{200}$? Ans. $10\frac{1}{2}$bush.

15. 32 is $\frac{8}{9}$ of how many times $\frac{1}{3}$ of 12? Ans. 9 times.

16. 36 is $\frac{4}{7}$ of how many times 9?

17. $\frac{2}{7}$ of 28 is $\frac{4}{9}$ of how many fifths of 15?

18. $\frac{5}{3}$ of 15 is $\frac{5}{11}$ of how many eighths of 88?

19. What is the value of $11\frac{1}{2}$yd. silk at $\frac{5}{4}$ of a dollar per yd.?

20. Sold a watch for $\$87\frac{1}{2}$, which was $\frac{7}{8}$ of its cost; what was lost by the transaction?

21. Reduce $\frac{2160}{4752}$ to its lowest terms. Ans. $\frac{5}{11}$.

22. How many cubic feet are there in $\frac{9}{11}$ of a cord of wood, and what is its value at $\$5\frac{1}{2}$ per cord? Ans. $104\frac{8}{11}$ ft.; $\$4\frac{1}{2}$.

23. Bought $73\frac{4}{9}$bush. corn for $\$64\frac{12}{72}$; what is the value of 64bush. at the same price? Ans. $56.

24. Reduce $\frac{2}{3}$ of a farthing to the fraction of a pound.

Ans. $\frac{1}{1440}$.

25. What is the value of $72\frac{1}{5}$lb. plums, at $27\frac{3}{4}$ cents?

Ans. $2003\frac{11}{20}$cts. $= \$20.03\frac{11}{20}$. (34, Note 4.)

26. How many cubic feet in a box that is $6\frac{3}{8}$ft. long, $5\frac{7}{9}$ft. wide and $3\frac{1}{5}$ft. deep?

$$6\frac{3}{8} \times 5\frac{7}{9} \times 3\frac{1}{5}\ (77) = \frac{51}{8} \times \frac{52}{9} \times \frac{16}{5} = \frac{\overset{17}{\cancel{51}} \times 52 \times \overset{2}{\cancel{16}}}{\cancel{8} \times \underset{3}{\cancel{9}} \times 5} = \frac{1768}{15} = 117\tfrac{13}{15}, \text{ Ans.}$$

NOTE. — In solving problems, it is desirable to cancel (127, a, Note 2) as much as convenient.

27. How many dozen bottles containing $1\frac{3}{4}$ pints each are required to bottle 63 gallons of wine? Ans. 24.

28. What is the value of my farm, which contains $73\frac{1}{11}$ acres, worth $\$96\frac{1}{8}$ per acre? Ans. $\$7025.86\frac{4}{11}$.

29. What cost $1763\frac{1}{9}$lbs. sugar at $6\frac{1}{4}$cts.? Ans. $\$110.19\frac{4}{9}$.

30. If it costs $\$8\frac{7}{8}$ to carry 13cwt. 3qr. $5\frac{3}{14}$lb. $19\frac{1}{4}$ miles, how far may it be carried for $\$64\frac{11}{32}$? Ans. $139\frac{9}{16}$ miles.

31. How many square feet of boards will be required to make 3 dozen boxes whose inner dimensions shall be $3\frac{1}{2}$ feet in length and breadth and $2\frac{1}{4}$ feet in depth, the boards being 1 inch in thickness? Ans. 2129.

32. How many feet will be required to make 36 boxes whose *outer* dimensions are as expressed in the preceding example, the boards being of the same thickness; and what is the difference of the capacities of the two sets of boxes in cubic inches?
Ans. 1907ft.; 274608c. in.

33. If $2\frac{3}{4}$yds. cloth cost \$7.70, what will $\frac{3}{5}$ of $\frac{5}{6}$ of a yard cost?
Ans. $\$1\frac{2}{5}$.

34. Bought $\frac{3}{5}$ of a 12 acre lot and sold $\frac{1}{3}$ of the part purchased; how much had I remaining? Ans. $4\frac{4}{5}$ acres.

35. The trans-Atlantic telegraph (mentioned page 26, Ex. 69) is to extend from St. John's, Newfoundland, to Valencia, Ireland, 1640 miles in a straight line; to allow for deviations from a straight course, inequalities of the sea-bottom, etc., the cable is to be $1\frac{43}{82}$ as long as would be required for a straight line; the iron wires in each bundle are twisted together, and the bundles run spirally around the cable. Now, supposing it necessary to increase the length of the wire 1 foot in every 20, in consequence

of twisting the wires, and 1 foot in 24 because of the bundles' running spirally, what length of wire will be required for the cable?

Ans. $362031\frac{1}{4}$ miles.

36. The trans-Atlantic telegraph cable is to weigh 1 ton per mile; what will it weigh per foot?

37. If it require $3\frac{1}{4}$ bushels of oats to sow an acre, how many bushels will be required to sow $7\frac{1}{5}$ acres? Ans. $23\frac{2}{5}$.

38. If I pay $\$1\frac{2}{5}$ per gallon for oil, what shall I pay for $13\frac{1}{4}$ gallons? Ans. $\$18\frac{11}{20}$.

39. A pole $12\frac{1}{2}$ feet long casts a shadow $3\frac{1}{8}$ feet at 12 o'clock; what is the length of the shadow cast by a steeple $133\frac{1}{8}$ feet high, at the same time? Ans. $33\frac{9}{32}$ft.

40. If a pole $12\frac{1}{2}$ feet long casts a shadow $3\frac{1}{8}$ feet at 12 o'clock, what is the hight of a steeple that casts a shadow $33\frac{9}{32}$ feet at the same time?

41. Bought 7 yards of one kind of cloth and $3\frac{1}{2}$ times as many yards of another kind; for the former I paid $\frac{1}{8}$ as many dollars per yard as there were yards of the latter, and for the latter $\frac{3}{5}$ as much per yard as for the former. What was the price per yard of each and the cost of the whole?

Ans. $\begin{cases} \$3\frac{1}{16}, \text{ price of former.} \\ \$1\frac{67}{80}, \text{ price of latter.} \\ \$66\frac{73}{160}, \text{ total cost.} \end{cases}$

42. If the cargo of a ship be worth \$72000, and if $\frac{3}{7}$ of $\frac{1}{2}$ of $\frac{7}{9}$ of the cargo be worth $\frac{4}{5}$ of $\frac{7}{8}$ of $\frac{10}{14}$ of the ship, what is the value of the ship? Ans. \$24000.

43. Of the inhabitants of a certain town, $\frac{3}{8}$ are farmers, $\frac{2}{7}$ mechanics, $\frac{1}{5}$ manufacturers, $\frac{1}{9}$ students and professional men, and the remainder, numbering 142, are engaged in various occupations. What is the population of the town. Ans. 5040.

44. A certain room is $16\frac{1}{2}$ feet in length, 15 feet in width, and $8\frac{2}{3}$ feet in height. In this room are 4 windows, each 3 feet wide by $4\frac{1}{3}$ feet high, having 12 panes in each window; also, 4 doors, each $3\frac{1}{4}$ feet wide by 8 feet high; the base-boards are $\frac{3}{5}$ of a foot wide. A glazier furnishes and sets the glass for $15\frac{1}{3}$ cents per pane, a painter paints the doors, base and floor for $3\frac{1}{4}$ cents per

square foot, and a mason plasters the room for $22\frac{1}{2}$ cents per square yard. What is the cost of glazing, painting and plastering? Ans. \$34.94$\frac{5}{8}$.

45. What would be the cost of carpeting the room mentioned in Example 44, the carpet being a yard wide and costing \$1$\frac{1}{3}$ per yard? Ans. \$36$\frac{2}{3}$.

46. What is the cost of papering the above room, the paper costing $\frac{9}{20}$ of a dollar per roll, each roll containing $17\frac{1}{7}$ yards, the paper being $1\frac{3}{4}$ feet in width, the paper-hanger charging $\frac{1}{6}$ of a dollar per roll for hanging it? Ans. \$2$\frac{7}{15}$.

47. A merchant bought $48\frac{3}{4}$ lbs. butter of one customer, $28\frac{1}{2}$ of another, $25\frac{3}{16}$ of another, and $56\frac{5}{8}$ of another; how many pounds did he buy, and what is the value of the lot at $22\frac{1}{2}$ cents per pound? Ans. $158\frac{13}{16}$ lbs.; \$35.73$\frac{9}{32}$.

48. A purchased 22 lbs. sugar at $6\frac{1}{4}$ cents, 10 lbs. tea at $\frac{1}{3}$ of a dollar, 50 lbs. rice at $4\frac{1}{2}$ cents, 1 bbl. flour at \$6$\frac{1}{8}$ and 38 yards of sheeting, and gave 2 ten-dollar bills to the merchant, who returned \$3$\frac{3}{4}$; required the price per yard of the sheeting?

Ans. $8\frac{1}{3}$cts.

49. How many times will a wheel that is $9\frac{1}{3}$ feet in circumference, turn round in running $17\frac{3}{4}$ miles? Ans. $10041\frac{3}{7}$.

50. The distance from the earth to the sun is about 95000000 miles; in what time will a car run that distance, running $37\frac{1}{2}$ miles per hour, allowing $365\frac{1}{4}$ days in a year?

Ans. 288y. 363d. 13h. 20m.

51. A farmer, owning $142\frac{1}{4}$ acres of land, sold 53a. 3r. 20rd.; how much had he left? Ans. $88\frac{3}{8}$a.

52. A farmer, owning $144\frac{1}{2}$ acres of land, cultivated $2\frac{1}{2}$ acres potatoes, $3\frac{1}{4}$ corn, $3\frac{1}{2}$ wheat, $1\frac{3}{4}$ rye and $1\frac{3}{4}$ oats, from which he harvested 250 bushels potatoes, $56\frac{1}{4}$ corn, $32\frac{1}{4}$ wheat, 25 rye and $62\frac{1}{2}$ oats per acre, respectively. He also cut $2\frac{1}{4}$ tons hay on each of $20\frac{1}{2}$ acres, and, upon the remainder of his land, he pastured 56 sheep, 12 cows, 2 pairs oxen and 3 horses for 25 weeks. He sold his potatoes, corn, wheat, rye and oats at $62\frac{1}{2}$, $87\frac{1}{2}$, $162\frac{1}{2}$, 100 and 45 cents per bushel, respectively; he sold his hay at \$12$\frac{1}{2}$ per ton, and received 4, 25, 40 and 50 cents each per

week, respectively, for pasturing the sheep, cows, oxen and horses. What were the net profits of this farm, supposing that he paid \$32 taxes, and that the cost of cultivating and harvesting the potatoes, corn, wheat, rye, oats and hay was \$35, \$35, \$33, \$25, \$15 and \$6 per acre, respectively? Ans. \$1070.28$\frac{29}{32}$.

53. A merchant, owning $\frac{3}{8}$ of a ship, sold $\frac{4}{6}$ of his share for \$3000; what was the value of the ship? Ans. \$12000.

54. The cargo of a certain ship is worth \$48000, and $\frac{5}{6}$ of the value of the cargo is $\frac{10}{3}$ the value of the ship; what is the ship worth? Ans. \$12000.

55. In a certain school $\frac{1}{2}$ the scholars study arithmetic, $\frac{1}{4}$ algebra, $\frac{1}{5}$ geometry, and the remainder of the school, viz. 10 scholars, study surveying; how many scholars are there in the school? Ans. 200.

56. $(\frac{3}{4}+\frac{5}{7}+\frac{8}{9}-\frac{2}{3}+\frac{1}{5}+\frac{1}{2}) \times 5\frac{1}{4}=?$ Ans. $12\frac{127}{240}$.

57. $\frac{3}{4}+\frac{5}{7}+(\frac{8}{9}-\frac{2}{3}+\frac{1}{5}+\frac{1}{2}) \times 5\frac{1}{4}=?$ Ans. $6\frac{257}{840}$.

58. $\frac{8}{13} \div 2+\frac{5}{12} \times 4-\frac{1}{9} \div \frac{1}{3}-\frac{2}{5} \times \frac{3}{2}=?$ Ans. $1\frac{8}{195}$.

59. $\frac{4}{7} \div (2+\frac{1}{2})-\frac{8}{7} \div 12=?$ Ans. $\frac{2}{15}$.

60. $\frac{3}{4}$ of $\dfrac{7\frac{1}{8}}{14\frac{1}{4}}+\dfrac{5\frac{4}{7}}{7\frac{4}{5}} \div 1\frac{3}{7}+(\frac{1}{5}$ of $\frac{5}{7}+\frac{1}{2} \times \frac{3}{2}) \div \frac{5}{8}-\dfrac{\frac{3}{8}}{3} \times 8=?$ Ans. $1\frac{17}{56}$.

61. £$\frac{3}{8}+\frac{4}{7}$s. $+\frac{1}{5}$d. $-\frac{1}{4}$qr. $=$ what fraction of a pound? what fraction of a shilling? penny? farthing?

1st. Ans. £$\frac{54317}{134400}$.

62. $27\frac{3}{4}$ is a divisor, and $5\frac{1}{3}$ is the quotient; what is the dividend? Ans. 148.

63. The sum of two numbers is $87\frac{1}{5}$, and one of the numbers is $18\frac{3}{4}$; what is the other? Ans. $68\frac{9}{20}$.

64. The difference of two numbers is $17\frac{2}{3}$, and the less number is $18\frac{1}{4}$; what is the greater? Ans. $35\frac{11}{12}$.

65. $48\frac{4}{5}$ is a dividend, and $24\frac{2}{5}$ is the quotient; what is the divisor. Ans. 2.

66. $47\frac{3}{5}$ is the product of two factors, and $12\frac{1}{3}$ is one of those factors; what is the other? Ans. $3\frac{159}{185}$.

67. $17\frac{2}{3}$ is a dividend, and $15\frac{3}{5}$ is the divisor; what is the quotient? Ans. $1\frac{31}{234}$.

68. The factors of a certain number are $32\frac{1}{4}$, $15\frac{1}{7}$ and $19\frac{4}{5}$; what is $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{4}{6}$ of the number? Ans. $3223\frac{11}{70}$.

69. What is the value of $\frac{2}{3}$ of a barrel of flour at $\$7\frac{4}{5}$ per barrel? Ans. $\$5\frac{1}{5}$.

70. How much cloth that is $\frac{3}{5}$ of a yard wide will it take to line a cloak containing $8\frac{1}{4}$ yards which is $\frac{11}{12}$ of a yard wide?

Ans. $12\frac{29}{48}$ yds.

71. A can build $33\frac{1}{3}$ rods of wall in $24\frac{1}{2}$ days by laboring $12\frac{1}{2}$ hours per day; in how many days of $9\frac{4}{5}$ hours will he build $1\frac{1}{2}$ times as many rods?

72. A garden whose breadth is 10 rods, and whose length is $1\frac{3}{5}$ times its breadth, has a wall $3\frac{1}{2}$ feet thick around it; what was the cost of digging a trench $2\frac{3}{4}$ feet deep, in which to lay this wall, at $\frac{3}{4}$ of a cent per cubic foot? Ans. $\$62.94\frac{3}{4}$.

73. What will be the cost of digging a ditch around the above-mentioned garden, within and adjacent to the wall, $3\frac{1}{2}$ feet wide and $2\frac{3}{4}$ feet deep, at $\frac{3}{8}$ of a cent per cubic foot?

74. A can perform a piece of work in 6 days of 10 hours each, and B can do the same in 8 days of 11 hours; in how many days of 11 hours can A and B together do the work?

Ans. $3\frac{9}{37}$.

75. A sold an ox for \$62.50, and received in payment $12\frac{1}{2}$ yards of broadcloth at $\$3\frac{1}{4}$ per yard and the balance in sugar at $12\frac{1}{2}$ cents per pound; how many pounds did he receive?

76. Bought a pair of oxen and a horse for \$340 and a wagon for $\frac{3}{8}$ the price of the horse. The oxen cost $\frac{9}{8}$ the price of the horse; what was the cost of each?

77. From a piece of land that is $7\frac{3}{8}$ rods long and $7\frac{1}{4}$ rods wide, take $3\frac{1}{2}$ square rods and $3\frac{1}{2}$ rods square, and what will remain? Ans. $37\frac{23}{32}$ square rods.

78. A owns $\frac{8}{13}$ of a field, and B, the remainder; the difference between their shares is 7a. 3r. $15\frac{1}{2}$rd. What is B's share?

79. A boy, having a number of marbles, gives to one schoolfellow $\frac{1}{5}$ of them; to another $\frac{1}{8}$ of the remainder; loses $\frac{1}{4}$ of what then remains; and sells $2\frac{1}{7}$ times as many as he loses, when he has but 6 marbles left. How many had he at first?

80. If a family of 5 persons eat $\frac{3}{5}$ of a barrel of flour in $4\frac{3}{5}$ weeks, how much will be sufficient for $24\frac{3}{8}$ weeks, if the family is increased by $\frac{3}{5}$ its former number?

81. A regiment of 1024 men are to be clothed with cloth that is $\frac{13}{8}$ of a yard wide. Now, if it takes $2\frac{3}{5}$ yards of this cloth for each soldier, how many yards of cloth that is $\frac{7}{8}$ of a yard wide will be sufficient to line all the garments?

82. What number is that which, being increased by $\frac{2}{3}$ of $\frac{3}{5}$ of $10\frac{5}{7}$ and the sum diminished by $7\frac{1}{2}$, will give a remainder of $9\frac{3}{7}$?

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§ 11. DECIMAL FRACTIONS.

141. A DECIMAL* FRACTION is a fraction whose denominator is a unit with one or more ciphers annexed.

142. The denominator of a *Vulgar Fraction* may be *any number whatever, and the* FORM OF THE DENOMINATOR *of a Decimal Fraction is its* DISTINGUISHING CHARACTERISTIC.

143. Every principle and every operation in Vulgar Fractions is equally applicable to Decimals;† but the peculiar form of the denominator gives facilities for operating in Decimals that do not exist in Vulgar Fractions.

144. The denominator of a decimal fraction is not usually expressed, since it can be easily determined, it being 1 with as many ciphers annexed as there are figures in the given decimal

145. A decimal fraction is distinguished from a whole number by a *point*, called the *decimal point* or *separatrix*, placed before the decimal; the first figure at the right of the point is *tenths;* the second, *hundredths;* the third, *thousandths;* etc.; thus, $.6 = \frac{6}{10}$, $.25 = \frac{25}{100}$, $.042 = \frac{42}{1000}$.

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* *Decimal,* from the Latin *decem, ten.*

† By the term *decimal* we usually mean a DECIMAL FRACTION.

146. Since whole numbers and decimal fractions both decrease by the same law from left to right, they may be expressed together in the same example, and numerated as in the following

NUMERATION TABLE.

Thousands,	Hundreds,	Tens,	Units,		Tenths,	Hundredths,	Thousandths,	Tenths of Thousandths,	Hundredths of Thousandths,	Millionths,	Tenths of Millionths,	Hundredths of Millionths,	Billionths,	Tenths of Billionths,	Hundredths of Billionths,	Trillionths,	Tenths of Trillionths,	Etc., etc.
3	4	7	1	.	6	5	9	8	7	2	8	4	3	2	1	6	5	

147. The integral number is numerated from the separatrix toward the *left*, and the fraction from the same point toward the *right*, each figure, both in the whole number and decimal, taking its name and value by its distance from the decimal point.

148. In reading a decimal, we may give the name to each figure separately, or read it as a whole number and give the name of the right hand figure only; thus, the expression .23 may be read $\frac{2}{10}$ and $\frac{3}{100}$, or it may be read $\frac{23}{100}$, for $\frac{2}{10}$ and $\frac{3}{100}$ $= \frac{20}{100}$ and $\frac{3}{100} = \frac{23}{100}$.

149. Since multiplying both terms of a fraction by the same number does not alter its value (133, a, Note 1), *annexing* one or more ciphers to a decimal does not affect its value; thus, $\frac{2}{10} = \frac{20}{100} = \frac{200}{1000}$, etc.; i. e. $.2 = .20 = .200$, etc.

150. *Prefixing* a cipher to a decimal, i. e. inserting a cipher between the separatrix and a decimal figure, *diminishes* the value of that figure to $\frac{1}{10}$ its previous value; for it removes the figure one place farther from the decimal point (147); thus, $.3 = \frac{3}{10}$, but $.03 =$ only $\frac{3}{100}$, which is but $\frac{1}{10}$ of $\frac{3}{10}$.

What is the effect of prefixing 2, 3 or more ciphers to a decimal?

151. A vulgar fraction is sometimes annexed to a decimal; thus, $.2\frac{1}{4}$. This is equivalent to the complex fraction $\frac{2\frac{1}{4}}{10}$. The vulgar fraction is never to be counted as a decimal place, but it is always a fraction of a unit of that order represented by the preceding decimal figure; thus, in $.234\frac{1}{2}$, the $\frac{1}{2}$ is half of *a thousandth.*

Notation of Decimal Fractions.

152. Let the pupil express in figures the following numbers:—

1. Twenty-seven hundredths. Ans. .27.
2. Thirteen thousandths.
3. Eighty-nine tenths of millionths.

Note.—An ambiguity often arises in enunciating a whole number and a decimal in the same example; thus, .203 is two hundred and three thousandths, and 200.003 is two hundred, and three thousandths. This ambiguity may, however, be avoided by placing the word *decimal* before the fraction; thus, 200.003 may be read two hundred and *decimal* three thousandths.

4. Write the decimal two hundred and fifty-two thousandths.
5. Decimal six hundred and sixty-three tenths of thousandths.
6. Five hundred and decimal five thousandths.
7. Three thousand and decimal three thousandths.
8. Twelve hundred and fifty and six-tenths.
9. Decimal seven hundred and seventy-seven thousandths.
10. Eight thousand and decimal eighteen millionths.

153. Let the following numbers be written in words, or read orally:—

1.	865.0004	6.	87654.00002
2.	42.4247	7.	40000.000004
3.	500.0005	8.	278.46943827
4.	796.6704	9.	202.4
5.	4.0000(06	10.	99.999999

NOTE 1.—Addition, subtraction, multiplication and division of decimal fractions are performed precisely as the same operations in whole numbers —no further explanation being necessary except to determine the place of the separatrix in the several results.

NOTE 2.—The *proofs* are the same as in whole numbers.

CASE 1.

154. To add decimal fractions,

RULE. — *Place tenths under tenths, hundredths under hundredths, etc.; then add as in whole numbers, and place the point in the sum directly under the points in the numbers added.*

	Ex. 1.	2.	3.
	4.56	80.7423	8742.94287
	87.942	5.9872	403.024
Sum,	92.502	86.7295	9145.96687
Proof,	92.502	86.7295	

4.	5.
876934.9427	3769.45270086534
37695.486794	27.006467892
769400.870064	.542
387645.000006	72.842761
427.06054	567842.047697387

6. Add 3.58, 647.2, 984.00087 and 2.46987.

7. Add 4869.5, 47.6908, 4.00306 and .87428.

8. Add 5678, 42.7, 98732.004 and .000006.

9. Add .569, .874, .5369, .8769432723.

10. Add 38.38, 5000.005, 300.003 and 33.333.

11. Add two hundred and decimal two thousandths; thirty-five millions and four millionths; thirteen thousandths; thirteen; forty thousandths; and decimal three hundred and three thousandths. Ans. 35000213.358004.

12. What is the sum of one hundred and fifty-three thousand four hundred and forty-seven, and sixteen millionths; fourteen, and four tenths of thousandths; five hundred and ten, and five hundredths of billionths; and decimal one hundred and seventy-seven thousandths?

Case 2.

155. To subtract a less decimal from a greater,

Rule.—*Place the less number under the greater, tenths under tenths, etc.; then subtract as in whole numbers, and place the point in the remainder, directly under the points in the minuend and subtrahend.*

	Ex. 1.	2.	3.
From	8.7 4 2 6	4 3.0 2 6 5 4	8 7.4 6 9 2
Take	3.8 6 1 4	3 2.6 9 8 4 7	2 7.2 3 5 4 8 7
Rem.	4.8 8 1 2	1 0.3 2 8 0 7	6 0.2 3 3 7 1 3
Proof,	8.7 4 2 6	4 3.0 2 6 5 4	

Note.—If, as in Ex. 3, there are more figures in the subtrahend than in the minuend, the deficiency may be supplied by annexing ciphers, or supposing them annexed, to the minuend.

4. From 876.54708 take 43.876952.

5. From 869542.7 take 32.57694287.

6. From seventy, and fourteen thousandths take sixteen, and sixteen hundredths.

7. From sixty-six millions take sixty-six millionths.

8. From .876954 take .00476954.

9. From 874369. take .534269.

10. From 3.0000542 take 1.47999.

Case 3.

156. To multiply one decimal fraction by another,

Rule.—*Multiply as in whole numbers, and point off as many figures for decimals in the product as there are decimal places in both factors, counted together.*

Ex. 1. Multiply .43 by .27.

	OPERATION.	PROOF.
Multiplicand,	.4 3	.2 7
Multiplier,	.2 7	.4 3
	3 0 1	8 1
	8 6	1 0 8
Product,	.1 1 6 1	.1 1 6 1

(a) If the number of figures in the product is less than the number of decimal places in the two factors, the deficiency must be supplied by prefixing ciphers to the product.

	2.	3.
Multiplicand,	34.5678	.25
Multiplier,	3.5	.25
	1728390	125
	1037034	50
Product,	120.98730	.0625

NOTE.—The reason of the rule for pointing the product will be obvious if we change the decimals to the form of vulgar fractions and then perform the multiplication; thus,

$.43 \times .27 = \frac{43}{100} \times \frac{27}{100} = \frac{1161}{10000} = .1161$, as in Ex. 1.
Again, $.25 \times .25 = \frac{25}{100} \times \frac{25}{100} = \frac{625}{10000} = .0625$, as in Ex. 3.

	4.	5.
Multiplicand,	.7284	.4786
Multiplier,	.00023	.27
	21852	33502
	14568	9572
Product,	.000167532	.129222

6. Multiply .4786 by .127. Ans. .0607822.
7. Multiply 587 by 4.32. Ans. 2535.84.
8. Multiply .427 by 345. Ans. 147.315.
9. Multiply 4.69 by 38.46.
10. Multiply .2467 by .1068.
11. Multiply 38.74 by 364.9.
12. Multiply .0008 by .0005.
13. Multiply 3874. by .2694.
14. Multiply 38.42 by 276.

(b) A decimal fraction may be multiplied by 10, 100, etc., by moving the separatrix as many places towards the *right* as there are ciphers in the multiplier; for by moving the point one place to the right, each figure in the multiplicand is made 10 times as great as before, and consequently the result is 10 times

as great as the multiplicand (147); thus, $87.45 \times 10 = 874.5$; $86.954 \times 100 = 8695.4$; $58.64 \times 10000 = 586400$.

15. Multiply 56.423 by 10.
16. Multiply 3467.28 by 10000000.
17. Multiply .0467 by 100. Ans. 4.67.
18. Multiply .00573 by 1000.
19. Multiply 376.94 by 1000.
20. Multiply 3.76 by 20. Ans. 75.2.

In Ex. 20, multiply by the factors of 20, viz., 10 and 2; i. e. move the point one place to the right and then multiply by 2.

21. Multiply 8.764 by 400. Ans. 3505.6.
22. Multiply 5.6432 by 38000.
23. Multiply .00004 by 56000.
24. Multiply 34.27 by 60000.
25. Multiply 8.469 by 804000.

Case 4.

157. To divide one decimal fraction by another,

Rule. — *Divide as in whole numbers, and point off as many figures for decimals in the quotient as the number of decimal places in the dividend exceeds those in the divisor.*

Ex. 1. Divide .645 by .15.

OPERATION.	PROOF.
.15).645(4.3	.15 Divisor.
60	4.3 Quotient.
45	45
45	60
0	.645 Dividend.

	2.	3.
Dividend,	8.43648108	1.2575125
Divisor,	.12	2.5
Quotient,	70.304009	.503005

(a) If the number of figures in the quotient is less than the excess of decimal places in the dividend over those of the divisor, supply the deficiency by prefixing ciphers to the quotient.

	4.	5.	6.
Dividend,	.0 0 0 7 9 2	.0 8 7 5 6	.1 2 4 8
Divisor,	.1 2	1.1	2.4
Quotient,	.0 0 6 6	.0 7 9 6	.0 5 2

NOTE. — The dividend is a product, of which the divisor and quotient are the factors (55); hence the rule for pointing the quotient.

7. Divide 38.7425 by .25. Ans. 154.97.
8. Divide 15.36246 by 469.8.
9. Divide .65084958 by 3.69.
10. Divide .176382 by .369.

(b) If there are more decimal places in the divisor than in the dividend, the number may be equalized by annexing one or more ciphers to the dividend. The quotient will then be a whole number.

11. Divide 1941.885 by .7846. Ans. 2475.
12. Divide 10634.16 by .4506.

(c) If there is a remainder after all the figures of the dividend have been used, the division may be continued by annexing ciphers to the dividend. Each cipher annexed becomes a decimal place in the dividend.

In some examples this operation may be continued until there is no remainder, but in others there will *necessarily* be a remainder, however far the operation may be continued. This latter class of examples gives rise to *circulating decimals*, which will be discussed in the Supplement. It may be remarked, however, that, if the divisor contains no prime factors but 2's and 5's, the division can *always* be carried until there shall be no remainder; but if there is any other prime factor in the divisor, the division *can never be completed* unless the *same* other factor is in the original dividend; for a dividend is not divisible by a divisor

unless it contains *all* the factors of the divisor; whereas annexing ciphers to the dividend introduces no prime factor into it, except 2's and 5's.

13. Divide .8746 by .32. Ans. 2.733125.
14. Divide .45 by .8.
15. Divide .87693 by .64.

NOTE.—When a decimal is not complete, we sometimes place the sign + after it, signifying that there is a remainder.

16. Divide .8742 by .56. Ans. 1.5610714 +.
17. Divide .34 by .27.
18. Divide 56.7 by 2.9.
19. Divide 87.69 by 47.
20. Divide 87.69 by .47.

(d) A decimal fraction may be divided by 10, 100, etc., by moving the separatrix as many places towards the *left* as there are ciphers in the divisor; for, by moving the point one place to the left, each figure in the dividend is made only $\frac{1}{10}$ as great as before, and consequently the result is only $\frac{1}{10}$ as great as the dividend (147); thus, $874.5 \div 10 = 87.45$; $8695.4 \div 100 = 86.954$; $46.87 \div 100000 = .0004687$.

21. Divide 7846.987 by 1000. Ans. 7.846987.
22. Divide 54.276 by 100000.
23. Divide 46.08 by 1000.
24. Divide .7842 by 1000.
25. Divide 769.428 by 200. Ans. 3.84714.

In Ex. 25, divide by the factors of 200, viz., 100 and 2; i. e., move the point two places to the left and then divide by 2.

26. Divide 48.9632 by 4000. Ans. .0122408.
27. Divide 769.842 by 3200.
28. Divide 3505.6 by 400.
29. Divide 874.69 by 64000.
30. Divide 46.8742 by 16000000.

Case 5.

158. $\frac{3}{4} \times 100 = \frac{300}{4} = 75$; and $75 \div 100 = .75$.

If a number be multiplied by any number, and the product be divided by the multiplier, the quotient will be the multiplicand (60). Now, in the above example, $\frac{3}{4}$ is multiplied by 100 by annexing two ciphers to the numerator; the fraction $\frac{300}{4}$ is then reduced to the whole number 75, and, finally, 75 is divided by 100 by placing the decimal point before the 75; $\therefore \frac{3}{4} = .75$. Hence,

To reduce a vulgar fraction to a decimal,

Rule.—*Annex one or more ciphers to the numerator and divide the result by the denominator, continuing the operation until there is no remainder, or as far as is desirable. Point off as many decimal places in the quotient as there are ciphers annexed to the numerator.*

Ex. 1. Reduce $\frac{5}{8}$ to a decimal fraction.

$\frac{5}{8} \times 1000 = \frac{5000}{8} = 625$; and $625 \div 1000 = .625$, Ans.

2. Reduce $\frac{3}{16}$ to a decimal. Ans. .1875.

3. Reduce $\frac{17}{32}$ to a decimal.

4. Reduce $\frac{5}{24}$ to a decimal. Ans. .20833+.

There being the factor 3 in the divisor, in Ex. 4, and no such element in the dividend, there must necessarily be a remainder, however far the division may be continued (157, c).

5. Reduce $\frac{7}{17}$ to a decimal.

6. Reduce $\frac{1}{2}$, $\frac{3}{5}$, $\frac{5}{64}$, $\frac{3}{25}$, $\frac{97}{153}$, $\frac{14}{29}$ and $\frac{456}{987}$ to decimals.

159. *Every decimal fraction is a vulgar fraction*, and, if its denominator be written, it will *appear* as such. It may then be reduced to lower terms, or modified like any other vulgar fraction.

Ex. 1. Reduce .24 to the form of a vulgar fraction and then to its lowest terms. $.24 = \frac{24}{100} = \frac{12}{50} = \frac{6}{25}$, Ans.

This process proves the rule in Art. 158.

2. Reduce .025.

$$.025 = \frac{25}{1000} = \frac{5}{200} = \frac{1}{40}, \text{ Ans.}$$

3. Reduce .13. Ans. $\frac{13}{100}$.

4. Reduce .4765, .87698, .000476 and .0075.

5. Reduce 3.5. $3.5 = \frac{35}{10} = \frac{7}{2}$, Ans.

Case 6.

160. Reduce 6d. and 3qr. to the decimal of a shilling.

1st. 3qr. $= \frac{3}{4}$d. $=$ .75d.; $\therefore$ 6d. and 3qr. $=$ 6.75d.

2d. 6.75d. $= \frac{6.75}{12}$s. $=$ .5625s.

The principle is the same as in Art. 158. Hence,

To reduce whole numbers of lower denominations to the decimal of a higher denomination,

RULE.—*Having annexed one or more ciphers to the lowest denomination, divide by the number it takes of that denomination to make one of the next higher, and annex the quotient as a decimal to that next higher; then divide the result by the number it takes of* THIS *denomination to make one of the* NEXT *higher, and so continue till it is brought to the denomination required.*

Ex. 1. Reduce 8s. 5d. 1qr. to the decimal of a pound.

4	1.0 0 qr.
12	5.2 5 0 0 d.
20	8.4 3 7 5 0 s.
	£ .4 2 1 8 7 5, Ans.

1qr. $=$ .25d.; 5.25d. $=$ .4375s.; 8.4375s. $=$ £.421875, Ans.

2. Reduce 1ft. 3in. 2b. c. to the decimal of a yard.

3	2.0 0 0 0 0 0 0 b. c.
12	3.6 6 6 6 6 6 6 + in.
3	1.3 0 5 5 5 5 5 + ft.
	.4 3 5 1 8 5 1 + yd., Ans.

In this example there will be a remainder, however far the operation is carried.

3. Reduce 6oz. 18dwt. 15gr. to the decimal of a pound Troy weight. Ans. .577604166lb.+.

4. Reduce 8℥ 4ʒ 2℈ 5gr. to the decimal of a pound.

5. Reduce 12cwt. 3qr. 21lb. 8oz. 4dr. to the decimal of a ton.

6. Reduce 5yd. 2ft. 6in. to the decimal of a rod, long measure.

12	6.0 in.
3	2.5 0 0 ft.
$5\frac{1}{2}$	5.8 3 3 3 + yd.
2	2
11	1 1.6 6 6 6 + half yds.
	1.0 6 0 6 + rods, Ans.

Since one of the divisors, in this example, is $5\frac{1}{2}$, both divisor and dividend are reduced to halves. The feet and inches are more than a half yard; ∴ the sum of the given numbers is more than a rod.

7. Reduce 543a. 3r. 36rd. 25yd. 8 ft. 36in. to the decimal of a square mile.

8. Reduce 56ft. 1725in. to the decimal of a cord.

9. Reduce 3qr. 2na. 1in. to the decimal of a yard.

10. Reduce 3qt. 1pt. 3gi. to the decimal of a gallon.

11. Reduce 3pk. 7qt. 1pt. to the decimal of a bushel.

12. Reduce 3wk. 6d. 40m. 30sec. to the decimal of a lunar month.

13. Reduce 8s. 24° 50″ to the decimal of a circumference.

14. Reduce 19cwt. 8oz. to the decimal of a ton.

15. Reduce 3fur. 25rd. 2yd. 6in. 2b. c. to the decimal of a mile.

Case 7.

161. To reduce a decimal of a higher denomination to whole numbers of lower denominations,

Rule.—*Multiply the given decimal by the number it takes of the next lower denomination to make one of this higher, and place the separatrix as in multiplication of decimals; multiply the* DECIMAL PART *of this product by the number it takes of the* NEXT *lower denomination to make one of* THIS, *and so proceed as far as necessary. The several numbers at the left of the points will be the answer.*

Ex. 1. Reduce .421875£ to shillings, pence and farthings.

Ans. 8s. 5d. 1qr.

```
£.421875
      20
8.437500s.
     12
5.2500d.
    4
1.00qr.
```

This article is the reverse of Art. 160; ∴ first multiply by 20, because there will be 20 times as many shillings as pounds. For a like reason, multiply the *fractional part* of a shilling by 12, to reduce it to pence, etc. After having fixed the decimal point in the several products, *the ciphers at the* RIGHT *of the significant figures are disregarded.*

2. Reduce .9375 of a gallon to quarts, pints and gills.

Ans. 3qt. 1pt. 2gi.

3. Reduce .84 of a lunar month to weeks, etc.

Ans. 3w. 2d. 12h. 28m. 48sec.

4. Reduce .7694 of an acre to roods, etc.

Ans. 3r. 3rd. 3yd. 1ft. 45.216in.

5. Reduce .6543 of a mile to furlongs, etc.

Ans. 5fur. 9rd. 2yd. 0ft. 2in. 1+b.c.

6. Reduce .54324 of a pound Troy to ounces, etc.
7. Reduce .5769lb. to ℥, ʒ, etc.
8. Reduce .0876 of a ton to cwt., qr., etc.
9. Reduce .9876 of a mile to fur., ch., etc.
10. Reduce .4698 of a cord to c.ft., cu.ft., etc.
11. Reduce .8694 of a yard to qr., na., etc.
12. Reduce .7564 of a bushel to pk., etc.

162. Miscellaneous Examples in Decimal Fractions.

Ex. 1. Bought 14.75yds. sheeting at 14 cents per yd.; what was the cost of the piece? Ans. $2.065.

Note. — Decimal fractions are peculiarly adapted to operations in Federal Money, the denominations of which conform to the decimal notation. *The dollar is the unit, and dimes, cents and mills are tenths, hundredths and thousandths.*

2. Bought 20.5 tons of hay at $12.375 per ton; what was the cost of the whole? Ans. $253.687½.

3. What is the value of 67.75 acres of land at $62.50 per acre?

4. Paid $4234.375 for 67.75 acres of land; what was the price per acre?

5. Paid $4234.375 for a piece of land at $62.50 per acre; how many acres were bought?

6. Bought land at $62.50 per acre, and sold it again at $75 per acre, thereby making $846.875; how many acres were bought?

7. Bought 67.75 acres of land at $62.50 per acre, and sold the lot for $5081.25; was there a gain, or loss? how much total and per acre?

8. Bought 1bbl. flour at $12.50, 3bush. corn at $87\frac{1}{2}$cts., 24.5lbs. sugar at $8\frac{1}{2}$cts., 3gal. molasses at $37\frac{1}{2}$cts., 2lbs. tea at $62\frac{1}{2}$cts., 6lbs. coffee at 11cts., 15lbs. rice at $4\frac{1}{4}$cts. and 4lbs. butter at 22cts.; what was the cost of the whole? Ans. $21.76.

9. Bought 4 loads of hay which weighed 2t. 15cwt. $12\frac{1}{2}$lb., 1t. 19cwt. $6\frac{1}{4}$lb., 2t. 4cwt. $18\frac{3}{4}$lb. and 3t. at $15.875 per ton; what did the whole cost? Ans. 157.46\frac{1}{64}$.

10. Bought 100 sheep at $1.375, and sold them again at $1.875; what was the gain per head and total?

11. Bought 133.5yd. broadcloth at $3.25, and sold 33yd. of it at 3.33\frac{1}{3}$, 50yd. at $3.875 and the remainder at $3.60; how much was gained by the transactions?

12. What cost 13yd. 2qr. 3na. of cloth at $4.67 per ell French — the ell French being 6qr.? Ans. 42.61\frac{3}{8}$.

13. What would $7\frac{3}{8}$ bales of cotton cost, each bale weighing 6.375cwt., at $11.75 per cwt.?

14. Bought 356.25lb. wool at $37\frac{1}{2}$cts., which was manufactured into cloth at an expense of $62.50; at what price must it be sold to gain $37.50?

4 15. What cost 5625 feet of boards at $15.625 per thousand? Ans. $87.890625

3 16. What cost 43a. 3r. 25rd. of land at $62.875 per acre?

17. What cost 3t. 15cwt. 2qr. $12\frac{1}{2}$lb. coal at $9.75 per ton?

18. What would be the cost of building 43m. 7fur. 25rd. of railroad at \$9562.87½ per mile?

19. What will be the expense of papering a room that is 20 feet long, 15 feet wide and 8.5 feet high, a roll of paper being 8 yards in length and ⅝ of a yard in width, and costing 62½cts. per roll?

20. What is the value of .875 cwt. of coal at £2 3s. 6d. 1qr. per ton?

21. A piece of land is 63.5 rods long and 27.75 rods wide; what will it cost to wall it, at 87½cts. per rod?

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§ 12. COMPOUND ADDITION.

163. A compound number is composed of two or more denominations (64) which do not usually increase decimally from right to left; consequently, in adding the different denominations, we do not carry one for *ten, but for the number it takes of the particular denomination added, to make a unit of the next higher;* thus, in adding Sterling or English Money, we carry for 4, 12 and 20, because 4qr. make 1d., 12d. make 1s., and 20s. make £1.

164. The *principle* of procedure is precisely the same as in simple addition. Hence,

To add compound numbers,

RULE. — *Write the numbers so that each denomination shall occupy a separate column, the lowest denomination at the right and the others towards the left in the order of their values. Add the numbers in the lowest denomination, divide the amount by the number it takes of this denomination to make one of the next higher, set the remainder under the column, and carry the quotient to the next column. So proceed until all the columns are added.*

165. PROOF. — *The same as in Simple Addition* (34).

Ex. 1.

£.	s.	d.	qr.
18	13	11	3
12	6	9	1
33	19	10	2
27	14	2	1
36	17	3	3
2	5	11	1
131	18	0	3

The amount of the first column is 11qr. = 2d. 3qr. Upon the same principle as in simple addition, the 3qr. are set under the column of farthings, and the 2d. are added to the pence, making 48d. = 4s. 0d. etc.

NOTE 1. — In writing and adding the numbers of *a single denomination*, the rules of simple addition must be observed.

2.

£.	s.	d.	qr.
45	17	6	3
18	19	11	2
20	13	10	2
72	9	4	1
97	0	8	3
255	1	5	3

3.

£.	s.	d.	qr.
4	6	9	3
8	7	4	2
18	15	0	1
9	6	4	1
8	14	0	3

4.

	yd.	ft.	in.
	2	2	11
	4	2	4
	3	1	7
	5	0	6
rd.	4	2	7
3	$4\frac{1}{2}$	0	11
or 3	4	2	5

5.

lb.	oz.	dwt.	gr.
14	10	19	21
7	11	12	5
18	2	4	20
5	0	18	14

6.

fur.	rd.	yd.	ft.	in.	b.c.
1	5	3	2	10	1
2	4	4	2	4	2
3	6	5	0	6	2
1	3	4	2	7	0
7	21	$1\frac{1}{2}$	2	4	2
or 7	21	2	0	10	2

7.

yd.	qr.	na.	in.
37	2	1	2
15	3	3	1
8	2	2	2
56	1	0	1

8.

gal.	qt.	pt.	gi.
16	3	1	3
5	1	0	1
14	1	1	2
57	0	1	0

9.

a.	r.	rd.
63	7	39
18	4	16
11	5	20
93	2	1

10. What is the sum of 56a. 3r. 37rd. 30yd. 8ft. 72in., 87a. 2r. 25rd. 15yd. 7ft. 143in., 15a. 1r. 14rd. 27yd. 2ft. 17in. and 53a. 3r. 33rd. 20yd. 8ft. 100in.?

11. Add together 53m. 7fur. 7ch. 2rd. 10li. 2in., 256m. 3fur 7ch. 2rd. 20li. 2in., 14m. 3fur. 2ch. 1rd. 23li. 4in. and 8m. 3fur. 2ch. 1rd. 13li. 6in.

12. Bought 3 pieces of cloth, measuring 15yd. 3qr. 2na. 2in., 24yd. 2qr. 3na. 1in. and 17yd. 2qr. 1na. 1in.; how much did I buy?

13. Add 2circ. 11s. 29° 59′ 59″, 3circ. 10s. 18° 15′ 13″, 5circ. 3s. 18° 42′ 15″ and 4circ. 8s. 17° 40′ 3″ together.

14. A farmer raised in one field, 58bush. 3pk. 4qt. 1pt. of wheat; in another, 100bush. 1pk. 7qt. 1pt.; and in another, 75bush. 1pk. 1qt. 1pt. How much wheat did he raise in the three fields?

15. A planter sold cotton at various different times as follows: 3t. 19cwt. 3qr. 21lb., 4t. 3cwt. 3qr. 14lb., 2t. 17cwt. 2qr. 21lb., 14t. 19cwt. 1qr. 7lb., 3t. 2qr. 7lb., 4cwt. 3qr. 21lb. and 3t 14lb.; what did he sell in all?

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§ 13. COMPOUND SUBTRACTION.

166. The principle of compound subtraction is like that of simple subtraction. Hence,

To perform Compound Subtraction,

RULE. — 1. *Write the less quantity under the greater, arranging the denominations as in addition.*

2. *Beginning at the right, take each denomination of the subtrahend from the number above it, and set the remainder beneath.*

3. *If any number of the subtrahend is greater than the number above it, add to the upper number as many as it takes of that denomination to make one of the next higher, and take the subtrahend from the* SUM; *set down the remainder, and add* ONE *to the next denomination in the subtrahend.*

167. Proof. — *As in simple subtraction* (37).

Ex. 1. From £6 4s. 3d. 2qr. take £4 1s. 2d. 1qr.

	£.	s.	d.	qr.
Min.	6	4	3	2
Sub.	4	1	2	1
Rem.	2	3	1	1
Proof,	6	4	3	2

Only the 1st and 2d sections of the rule apply to this example.

2. From 6£ 9s. 11d. 1qr. take 3£ 15s. 5d. 3qr.

	£	s.	d.	qr.
Min.	6	9	11	1
Sub.	3	15	5	3
Rem.	2	14	5	2
Proof,	6	9	11	1

As 3qr. cannot be taken from 1qr., borrow one of the 11d., reduce it to qr. and add it to the 1qr., then say 3qr. from 5qr. leaves 2qr. Now, as one of the 11d. has been employed, say 5d. from 10d., or, what is practically the same, 6d. from 11d. leaves 5d., and so proceed through the example.

The plan and the reasoning are the same as in simple subtraction (36, Ex. 3), and the example may be solved as follows:—

	£	s.	d.	qr.		£	s.	d.	qr.
Min.	6	9	11	1	=	5	29	10	5
Sub.	3	15	5	3		3	15	5	3
Rem.	2	14	5	2	=	2	14	5	2

3.

	a.	r.	rd.	yd.	ft.	in.
From	8	3	30	30	8	133
Take	4	2	35	25	4	143
Rem.	4	0	35	5	3	134
Proof,	8	3	30	30	8	133

4.

	gal.	qt.	pt.	gi.
From	27	3	1	2
Take	14	2	1	3
Rem.	13	0	1	3
Proof,	27	3	1	2

5.

	t.	cwt.	qr.	lb.	oz.	dr
From	18	15	2	24	14	12
Take	15	19	3	20	12	14
Rem.						
Proof,						

6.

	lb.	oz.	dr.	sc.	gr.
From	15	6	4	2	14
Take	2	11	3	1	19
Rem.					
Proof,					

168. Sometimes, as in the following examples, it is neces sary to borrow *two* of the higher denomination of the minuend instead of *one;* but in all such cases we must carry *two* to the next term of the subtrahend; i. e. *we must* PAY *as much as we* BORROW.

7.

	rd.	yd.	ft.	in.	b. c.		rd.	yd.	ft.	in.	b. c.
From	12	0	2	6	1	=	10	10	4	17	4
Take	3	5	2	8	2		3	5	2	8	2
Rem.	7	5	2	9	2	=	7	5	2	9	2
Proof,	12	0	2	6	1	=	10	10	4	17	4

8.

	a.	r.	rd.	yd.	ft.	in.		a.	r.	rd.	yd.	ft.	in
From	12	2	0	0	6	128	=	11	4	78	60	10	200
Take	3	3	39	30	8	143		3	3	39	30	8	143
Rem.	8	1	39	$29\frac{1}{2}$	6	129	=	8	1	39	30	2	57
Proof,	12	2	0	0	6	128	=	11	4	78	60	10	200

9.

	m.	fur.	rd.	yd.	ft.
Min.	9	7	10	0	1
Sub.	2	6	4	5	2
Rem.					
Proof,					

10.

circ.	deg.	m.	fur.	rd.	yd.	ft.
5	0	0	3	0	0	1
2	27	69	5	39	5	2

169. The following examples are similar to the preceding, but the rule for subtraction is inapplicable until the *form* of the minuend or subtrahend is changed.

11.

	rd.	yd.	ft.	in.	b. c.		rd.	yd.	ft.	in	b. c.
From	15	0	0	0	0	=	14	5	1	5	3
Take	14	5	1	5	2		14	5	1	5	2
Rem.											1
Proof,							14	5	1	5	3

12.

	rd.	yd.	ft.	in.	b.c.		rd.	yd.	ft.	in.	b.c.
Min.	3	0	1	0	1	=	3	0	1	0	1
Sub.	2	5	2	5	2		3	0	0	11	2
Rem.											2
Proof,							3	0	1	0	1

13.

	a.	r.	rd.	yd.	ft.	in.
From	12	3	0	0	5	0
Take	12	2	39	30	6	36
Rem.						
Proof,						

14.

	circ.	deg.	m.	fur.	rd.	yd.	ft.	in.
From	10	0	0	0	0	0	0	0
Take	9	359	69	3	39	5	1	5
Rem.								
Proof,								

What are the peculiarities of these examples?

(a) Another way to solve such examples is to reduce both minuend and subtrahend to the lowest denomination contained in either (87), and then subtract as in simple subtraction.

170. In subtracting an earlier from a later date, it is customary to consider 30 days a month.

15. What is the difference in time between April 17, 1827, and February 12, 1834?

	yr.	m.	d.
Min.	1834	2	12
Sub.	1827	4	17
Rem.	6	9	25

16. Find the time from Aug. 15, 1843, to Dec. 12, 1851.
17. Find the time from May 12, 1841, to June 21, 1842.
18. Find the time from June 21, 1842, to Aug. 24, 1846.
19. Find the time from Aug. 24, 1846, to Sept. 1, 1847.
20. Find the time from July 21, 1836, to Sept. 1, 1847.
21. Find the time from Feb. 29, 1816, to Aug. 22, 1855.
22. Find the time from Jan. 8, 1743, to Dec. 16, 1854.
23. Find the time from Sept. 6, 1777, to Nov. 13, 1816

171. Examples in Compound Addition and Subtraction.

1. My farm contained 108a. 3r. 17rd., and I have sold to one man 15a. 2r. 29rd., and to another 11a. 33rd.; how much have I left?

2. Having a journey of 187m. 7fur. 19rd. to perform in 3 days, I travel 53m. 3fur. 14rd. the first day and 71m. 18rd. the second; how far must I travel on the third?

3. Bought 723 acres of land for 963£ 14s. 9d.; from this sold to A 253a. 3r. 17rd. for 319£ 12s., and to B 176a. 14rd. for 237£ 11d.; how much land remains, and what has it cost?

4. How long since the Declaration of American Independence, July 4, 1776?

5. A bought of one farmer 11t. 7cwt. 3qr. 18lb. of cheese; of another, 6t. 19cwt. 7lb.; of another, 18cwt. 3qr.; and of another, 17t. 3qr. He also sold in Boston, 12t. 7cwt. 12lb.; in Lowell, 8t. 15cwt. 3qr., and the remainder in New York. How much did he sell in New York?

6. B sold an ox which weighed 17cwt. 3qr. 8lb.; and 2 cows that weighed 5cwt. 3qr. 18lb. each; and 3 swine that weighed 3cwt, 2qr. 12 lb., 4cwt. 1qr. 18lb., and 5cwt. 3qr. 6lb. respectively. How much more beef than pork did he sell?

7. From the sum of 7rd. 2yd. 2in. 1b.c. and 2rd. 3yd. 1ft. 3in. 2b.c., take the difference between 14rd. 2ft. 7in. 1b. c. and 4rd. 2ft. 7in. 2b.c. Ans. 1b.c.

8. From a piece of cloth measuring 18yd. 3qr. 3na. 2in. there were cut 3 garments, the first measuring 4yd. 3na., the second 3yd. 2qr., and the third 5yd. 1qr. 2na. 1in. How much cloth remained?

9. If from 2 casks of molasses, containing 65gal. 3qt. 1pt. and 74gal. 1pt. 3gi., there be taken 83gal. 1qt., how many gallons, quarts, etc., will remain?

10. How long from the battle of Lexington, April 19, 1775, to the Declaration of Independence, July 4, 1776?

11. How long from the battle of Bunker Hill, June 17, 1775, to the erection of General Warren's Statue, June 17, 1857?

§ 14. COMPOUND MULTIPLICATION.

172. In compound multiplication, the multiplicand only is compound.

173. In both simple and compound multiplication, the *multiplier is always and necessarily an abstract number;* for, to attempt to multiply by a concrete number, e. g. 4 *miles* times 10, is, in the highest degree, absurd, though it is perfectly proper to say 10 times 4 miles.

174. The product is of the same kind as the multiplicand for repeating a number does not change its nature.

175. The principle is the same as in simple multiplication. Hence,

To multiply a Compound by a Simple Number,

RULE. — *Multiply the lowest denomination in the multiplicand, divide the product by the number it takes of that denomination to make one of the next higher, set down the remainder, carry the quotient to the product of the next denomination, and so proceed.*

Ex. 1.

	£.	s.	d.	qr.
Multiply	4	6	8	3
By				7
Product,	30	7	1	1

First, 7 times 3qr. = 21qr. = 5d. and 1qr.; write the 1qr. under the farthings, and then say 7 times 8d. = 56d., and 5d. added give 61d. = 5s. and 1d., etc.

NOTE. — Multiplication and division mutually prove each other. It is profitable to teach reverse operations simultaneously.

2.

	fur.	rd.	yd.	ft.	in.	b.c.
Multiplicand,		9	4	2	6	2
Multiplier,						7
Product,	1	29	0	2	10	2

3.

	gal.	qt.	pt.	gi.
Multiplicand,	2	3	1	2
Multiplier,				5
Product,	14	2	1	2

4.

	t.	cwt.	qr.	lb.	oz.	dr.
Multiplicand,	4	3	2	14	8	4
Multiplier,						11
Product,	46	0	0	9	10	12

5.

	a.	r.	rd.	yd.	ft.	
	5	3	37	25	8	
					12	
	71	3	14	$7\frac{1}{2}$	6	in.
or	71	3	14	8	1	72

176. When the multiplier is a composite number, we may proceed as in like cases in simple multiplication (44).

6. Multiply 4lb. 8oz. 16dwt. 20gr. by 72.

	lb.	oz.	dwt.	gr.
Multiplicand,	4	8	16	20
1st Factor of Multiplier,				8
Partial Product,	37	10	14	16
2d Factor of Multiplier,				9
Product,	341	0	12	0

7. Multiply 8lb. 4℥ 7ʒ 2℈ 16gr. by 63.
8. Multiply 9m. 7fur. 8ch. 3rd. 15li. 6in. by 96.

177. When the multiplier is large and not composite, some expedient may be adopted, as in the following examples.

9. Multiply 2bush. 3pk. 4qt. 1pt. by 47.

bush.	pk.	qt.	pt.	
2	3	4	1	Multiplicand.
			5	
14	1	6	1	= 5 times Multiplicand.
			9	
130	0	2	1	= 45 times Multiplicand.
5	3	1	0	= 2 times Multiplicand.
135	3	3	1	= 47 times Multiplicand.

First multiply by 45, i. e. by 5 and 9; then *add* twice the multiplicand, and thus multiply by 47.

This example may be solved as follows:—

bush	pk.	qt.	pt.	
2	3	4	1	Multiplicand.
			6	
17	1	3	0	= 6 times Multiplicand.
			8	
138	3	0	0	= 48 times Multiplicand.
2	3	4	1	= once Multiplicand.
135	3	3	1	= 47 times Multiplicand.

Here we multiply by 48, i. e. by 6 and 8; then *subtract* the multiplicand.

This plan may be indefinitely modified; hence this general direction: — Multiply by two or more numbers whose product is nearly the multiplier, and add to or subtract from the product such numbers as the case may require.

10. Multiply 27yd. 3qr. 2na. by 59.
11. Multiply 17gal. 3qt. 1pt. 3gi. by 97.
12. What is the cost of 857 yards of cloth, at 3£ 15s. 6d. 1qr. per yard?

£.	s.	d.	qr.	
3	15	6	1	
			10	
37	15	2	2	= cost of 10yd.
			10	
377	12	1	0	= cost of 100yd.
			8	
3020	16	8	0	= cost of 800yd.
188	16	0	2	= cost of 50yd.
26	8	7	3	= cost of 7yd.
3236	1	4	1	= cost of 857yd.

Multiply by 100, i. e. by 10 and 10; then multiply the cost of 100 yards by 8, the cost of 10 yards by 5 and the cost of 1 yard by 7, which will give the cost of 800, 50 and 7 yards, severally; finally, *add* the cost of 800, 50 and 7 yards together, and thus find the cost of 857 yards, the answer.

13. Bought 131 loads of wood, each measuring 1c. 3c.ft. 7cu.ft., at $5.67 per cord; what was the quantity bought, and the cost of the whole?

14. If a ship sail 2° 14′ 27″ per day, how far will she sail in 29 days?

15. If 3 men build 7rd. 5ft. of wall in 1 day, how much will they build in 17 days?

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§. 15. COMPOUND DIVISION.

178. Here, as in the three preceding sections, the principle is the same as in the corresponding operation in simple numbers. Hence,

To divide a Compound by a Simple Number,

RULE.—*Divide the highest denomination of the dividend, and set down the quotient; if there is a remainder, reduce it to the next lower denomination; to the result add the given quantity of that denomination and divide as before, setting down the quotient and reducing the remainder, etc.*

Ex. 1. Divide 30£ 7s. 1d. 1qr. by 7.

	£	s.	d.	qr.	
7)	30	7	1	1	
	4	6	8	3,	Ans.
				7	
	30	7	1	1,	Proof.

30£ ÷ 7 gives a quotient of 4£ and a remainder of 2£. 2£ reduced to shillings and added to 7s., give 47s., which, divided by 7, gives a quotient of 6s., and a remainder of 5s., etc., etc.

2. Divide 1fur. 29rd. 0yd. 2ft. 10in. 2b. c. by 7.
3. Divide 14gal. 2qt. 1pt. 2gi. by 5.
4. Divide 46t. 9lb. 10oz. 12dr. by 11.
5. Divide 71a. 3r. 14rd. 8yd. 1ft. 72in. by 12.

179. When the divisor is composite, we may divide by its factors, as in simple division (56).

6. Divide 341lb. 0oz. 12dwt. by 72.

$$\begin{array}{r|rrrr} & \text{lb.} & \text{oz.} & \text{dwt.} & \text{gr.} \\ 9) & 341 & 0 & 12 & 0 \\ \hline 8) & 37 & 10 & 14 & 16 \\ \hline & 4 & 8 & 16 & 20, \text{ Ans.} \end{array}$$

First divide by 9 and then the quotient by 8, and thus by 72.

7. Divide 530 lb. 2℥ 3ʒ 2℈ 8gr. by 63.

8. Divide 958m. 5fur. 5ch. 12li. $5\frac{12}{25}$in. by 96.

180. When the divisor is large and not composite, set down the work of dividing and reducing. There is no device for rendering the operation easier.

9 Divide 135bush, 3pk. 3qt. 1pt. by 47.

$$\begin{array}{rl} & \text{bush.}\ \ \text{pk.}\ \ \text{qt.}\ \ \text{pt.} \\ 47) & 135\quad 3\quad 3\quad 1\ (\text{2bush. 3pk. 4qt. 1 pt., Ans.} \\ & \underline{94} \\ & 41 \text{ bush.} \\ & \underline{\ \ 4} \\ & 167 \text{ pk.} \\ & \underline{141} \\ & 26 \text{ pk.} \\ & \underline{\ \ 8} \\ & 211 \text{ qt.} \\ & \underline{188} \\ & 23 \text{ qt.} \\ & \underline{\ \ 2} \\ & 47 \text{ pt.} \\ & \underline{47} \\ & 0 \end{array}$$

Having found that 47 is contained twice in 135, multiply 47 by 2, and subtract the product, 94, from 135, which leaves a remainder of 41 bushels; reduce the 41 bushels to pecks and add the 3 pecks, making 167 pecks; then divide by 47, and so continue the process till the work is done.

10. Divide 1644yd. 2qr. 2na. by 59.

11. Divide 1742gal. 3qt. 1pt. 3gi. by 97.

12. If 857 yards of cloth cost 3236£ 1s. 4d. 1qr., what is the price per yard?

13. Sold 131 equal loads of wood, measuring 187c. 2c. ft. 5cu. ft. for $\$1061.92\frac{115}{128}$; what was the quantity per load and the price per cord?

14. If, in 29 days, a ship sail 2s. 4° 59′ 3″, how far is that per day?

15. If 5 men build 124rd. 2ft. 6in. of wall in 17 days, how much would they build in one day?

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§ 16. DUODECIMALS.

181. Duodecimals * are compound numbers which decrease *uniformly* from the highest to the lowest denomination by the constant divisor, 12.

182. This measure is usually applied to feet and parts of a foot, and is used by artificers in determining distances, areas and solidities.

Its denominations are feet (ft.), inches (′), seconds (″), thirds (‴), fourths (⁗), etc., etc.

The accents used to designate the denominations are called *indices.*

183. The foot being the unit, the denominations have the relations indicated by the following

TABLE.

$1' =$ $\frac{1}{12}$ of a foot.
$1'' = \frac{1}{12}$ of $1' = \frac{1}{12}$ of $\frac{1}{12}$ of 1 ft. $= \frac{1}{144}$ of a foot.
$1''' = \frac{1}{12}$ of $1'' = \frac{1}{12}$ of $\frac{1}{144}$ of 1 ft. $= \frac{1}{1728}$ of a foot.
$1'''' = \frac{1}{12}$ of $1''' = \frac{1}{12}$ of $\frac{1}{1728}$ of 1 ft. $= \frac{1}{20736}$ of a foot.
etc. etc.

Thus 12 of any lower denomination make 1 of the next higher; e. g.

$$12'''' = 1'''$$
$$12''' = 1''$$
$$12'' = 1'$$
$$12' = 1\text{ft.}$$

---

* *Duodecimal,* from the Latin *duodecim, twelve.*

184. Addition, subtraction and division of duodecimals are performed as the like operations of other compound numbers; the same is true of multiplication, except that, when both factors are in the form of compound numbers, we are required *to determine the denomination of the product.*

185. In this investigation, *for the sake of convenience,* we familiarly speak of multiplying feet by feet, feet by inches, inches by inches, etc., though here, as everywhere (173), the multiplier is strictly an *abstract* number; e. g., suppose a board is 10 feet long and 1 foot wide, it evidently contains 10 square feet, and if it is 10 feet long and 2 feet wide, it as evidently contains 2 times 10 square feet $=$ 20 square feet, though it would be *nonsense* to affirm that it contains 2 *feet* times 10 feet; still, we are accustomed to say that the area of a board is equal to its *length multiplied* by its *breadth.* Again, if a board is 10 feet long and 1 inch wide, it contains $\frac{1}{12}$ as many *square feet* as it is *feet in length;* i. e. it contains $\frac{1}{12}$ of 10 square feet $= \frac{10}{12}$sq. ft. $= 10'$; and if the board is 10ft. long and 2in. wide, it contains $\frac{2}{12}$ of 10sq. ft. $= \frac{20}{12}$ of a sq. ft. $= 1\frac{8}{12}$sq. ft. $=$ 1ft. and $8'$. This illustration can be carried to any extent.

186. Since $1' = \frac{1}{12}$ft., $1'' = \frac{1}{144}$ft., $1''' = \frac{1}{1728}$ft., etc., whether the measure is linear, square or cubic, it follows that **1′**, in linear measure, is a *line,* $\frac{1}{12}$ of a foot in length; in square measure, $1'$ is an *area,* 1 foot long and 1 inch wide, and $1''$ is an *area,* 1 inch square; in cubic measure, $1'$ is a *solid,* 1 foot long, 1 foot wide and 1 inch deep, $1''$ is a *solid,* 1 foot long, 1 inch wide and 1 inch deep and $1'''$ is a *cubic inch;* etc.

187. Let us now proceed to determine the denomination of the product obtained by multiplying any two denominations together.

PHILOSOPHICALLY.	FAMILIARLY.
2 units $\times$ 3 units $=$ 6 units,	i. e. 2ft. $\times$ 3ft. $=$ 6ft.
2 " $\times \frac{3}{12}$ unit $= \frac{6}{12}$ unit,	i. e. 2ft. $\times 3' = 6'$.
2 " $\times \frac{3}{144}$ " $= \frac{6}{144}$ ",	i. e. 2ft. $\times 3'' = 6''$.
etc.	etc.

PHILOSOPHICALLY.	FAMILIARLY.
$\frac{2}{12}$ unit $\times$ $\frac{3}{12}$ unit $=$ $\frac{6}{144}$ unit,	i. e. $2' \times 3' = 6''$
$\frac{2}{12}$ " $\times$ $\frac{3}{144}$ " $=$ $\frac{6}{1728}$ " ,	i. e. $2' \times 3'' = 6'''$
$\frac{2}{12}$ " $\times$ $\frac{3}{1728}$ " $=$ $\frac{6}{20736}$ " ,	i. e. $2' \times 3''' = 6''''$
etc.	etc.
$\frac{2}{144}$ unit $\times$ $\frac{3}{144}$ unit $=$ $\frac{6}{20736}$ unit,	i. e. $2'' \times 3'' = 6''''$
$\frac{2}{144}$ " $\times$ $\frac{3}{1728}$ " $=$ $\frac{6}{248832}$ " ,	i. e. $2'' \times 3''' = 6'''''$
$\frac{2}{144}$ " $\times$ $\frac{3}{20736}$ " $=$ $\frac{6}{2985984}$ " ,	i. e. $2'' \times 3'''' = 6''''''$
etc.	etc.

Hence, to determine the denomination of the product of two factors in duodecimals,

RULE. — *Add the indices of the two factors together, and the sum will be the index of the product.*

Ex. 1. A board is 6ft. 7′ 9″ in length and 2ft. 7′ 5″ in breadth; what is its area?

	6	7′	9″		
	2	7′	5″		
	13	3′	6″		
	3	10′	6″	3‴	
		2′	9″	2‴	9⁗
Ans.	17	4′	9″	5‴	9⁗

First, 9″ × 2 = 18″ = 1′ 6″; the 6″ we write under the seconds, and reserve the 1′ to add to the next product, thus, 7′ × 2 = 14′, which increased by the 1′ previously obtained gives 15′ = 1ft. 3′; the 3′ is written down, and the 1ft. is carried to the product of the feet, making 13ft. In like manner we multiply the multiplicand by the 7′ and then by the 5″, setting the partial products as in the margin. Finally, the *sum* of these partial products is the product sought. Hence,

188. To perform Multiplication of Duodecimals,

RULE. — *By the rule in compound multiplication, multiply each term in the multiplicand by each in the multiplier, and write the terms of the several partial products in the order of their values, so that similar terms shall stand in a column together; the sum of the partial products will be the entire product.*

Ex. 2. What quantity of boards will be required to lay a floor 14ft. 8′ 3″ in length and 13ft. 6′ 9″ in width?

Ans. 199ft. 2′ 4″ 8‴ 3⁗.

3. What are the contents of a granite block that is 8ft. 9′ 3″ long, 3ft. 2′ 4″ wide and 2ft. 5′ 7″ thick?

Ans. 69ft. 0′ 10″ 4‴ 5⁗ 1′′′′′.

4. How many bricks are required to build a wall 40ft. 8′ in length, 15ft. 6′ in hight and 1ft. 4′ in thickness, the dimensions of a brick being 8′, 4′ and 2′?

5. The walls of a house are 1ft. 4′ thick; how many bricks were required to build it, the house being 38ft. long, 26ft. 8′ wide and 18ft. 6′ high?

6. A certain house has 4 tiers of windows and 11 windows in each tier; the hight of the first tier is 5ft. 8′, of the second 5ft. 4′, of the third 4ft. 9′ and of the fourth 4ft. 3′; the breadth of each window is 3ft. 9′. How many square feet do they contain?

189. The merchantable thickness of boards is 1′; ∴, to reduce timber, joist, plank, etc. to board measure, *find the area of one face of the stick, and multiply this by the number expressing its thickness in inches.*

7. How many feet, board measure, in a plank 12ft. 4′ long, 2ft. 3′ wide and 4′ thick? Ans. 111.

8. How many feet, board measure, in a plank 40ft. 6′ long, 2ft. 6′ wide and 2′ 9″ thick? Ans. 278ft. 5′ 3″.

9. How many feet, board measure, in a stick of timber 36ft. 9′ long, 9′ wide and 6′ 6″ thick?

10. How many feet of boards will be required to make 12 boxes whose interior dimensions are 5ft. 6′, 4ft. 9′ and 3ft. 8′, the boards being 1′ in thickness?

11. How many feet less are required to make 12 boxes whose *exterior* dimensions are like the *interior* of those in Ex. 10, the boards being of the same thickness? Ans. 111ft. 4′.

12. What is the difference of the capacities of the two sets of boxes described in Ex. 10 and 11? Ans. 122ft. 10′.

13. How many square feet in the floor, ceiling and four walls of a room that is 18ft. 6′ long, 15ft. 9′ wide and 8ft. 4′ high?

14. How many square yards of carpet will be required to cover a floor that is 16ft. 6′ long and 15ft. 8′ wide?

15. How many cubic feet in a cellar that is 38ft. 6′ long, 24ft. 8′ wide and 7ft. 9′ deep?

16. How many cubic yards of earth must be removed in digging a cellar 40ft. 6′ in length and 24ft. in width, between the walls, the walls to be 2ft. 6′ in thickness, and the cellar to be 6ft. 6′ in depth from the surface of the ground?

17. A pile of wood is 8ft. long, 6ft. high and 4ft. wide; how many cords does it contain?

NOTE.—The 17th Ex. may evidently be solved by multiplying the length, breadth and hight together, and dividing the product by 128; thus,

$$\frac{\cancel{8} \times \overset{3}{\cancel{6}} \times \cancel{4}}{\cancel{128}\ \cancel{32}\ \cancel{4}\ 2} = \frac{3}{2} = 1\tfrac{1}{2} \text{ cords, Ans.};$$

but, wood for market is usually cut 4 feet in length, and, consequently, we may find the number of cords in a pile of such wood by multiplying the length of the pile by its hight and dividing the product by 32; thus, in the last example,

$$\frac{\cancel{8} \times 6}{\cancel{32}\ \ 4} = \frac{3}{2} = 1\tfrac{1}{2} \text{ cords, Ans. as before.}$$

18. How many cords of wood in a pile that is 20ft. 6′ long, 8ft. high and 4ft. wide?

19. How many cords in a pile that is 67ft. 9′ long, 17ft. 3′ high and 4ft. wide?

20. How many cords in a pile that is 25ft. long, 7ft. high and 4ft. wide?

21. How many cords in a pile that is 33ft. 8′ long, 6ft. 6′ high and 3ft. 10′ wide?

§ 17. INTEREST.

190. When money is lent, the borrower pays the lender *for its use.*

191. The money lent is called the *Principal;* the sum paid *for the use* of the principal is the *Interest;* the *sum* of the *principal* and *interest* is the *Amount.*

192. The interest is a certain *per cent.* * of the *principal;* i. e. so many dollars for each *hundred* dollars, so many pounds for each *hundred* pounds, etc.; e. g. \$6 for \$100, £6 for £100, etc., is 6 per cent.; \$5 for \$100, 5cts. for 100cts., etc., is 5 per cent.

The *annual* per centage is the *Rate.*

193. The *rate* is usually *fixed by law.*

In New England and most of the United States the *legal* or *lawful* rate is 6 per cent.; in New York, 7 per cent.; in Illinois and most of the Western States, as high as 10 per cent. by agreement; in Texas, as high as 12 per cent.; in California, *any* rate by agreement, etc. In France and England, 5 per cent.

Note.—In this treatise, 6 per cent. is understood when no per cent. is mentioned.

194. When the rate is 6 per cent., the interest of \$1 for a year is 6cts.; for 2 years, 12cts., etc.; for 1 month, $\frac{1}{12}$ of 6cts. $=$ 5 mills or $\frac{1}{2}$ct.; for 2 months, 1ct.; 3 months, $1\frac{1}{2}$cts.; 6 months, 3cts.; 9 months, $4\frac{1}{2}$cts., etc.; for 1 day, $\frac{1}{30}$ of 5 mills $= \frac{1}{6}$ mill; for 2 days, $\frac{1}{3}$m.; 3 days, $\frac{1}{2}$m.; 4 days, $\frac{2}{3}$m.; 5 days, $\frac{5}{6}$m.; 6 days, 1m.; 7 days, $1\frac{1}{6}$m.; 9 days, $1\frac{1}{2}$m.; 12 days, 2m.; 24 days, 4m.; etc., etc. Hence,

To find the interest of \$1 at 6 per cent., for any time,

---

* *Per cent.* is a contraction of *per centum,* a Latin phrase which means *by the hundred.*

RULE.—*Take* 6*cts.* (= $.06) *for each year,* 1*ct. for each* 2 *months in the part of a year,* 5 *mills* (= $.005) *for the odd month, if there be one, and* 1 *mill for each* 6 *days in the part of a month.*

Ex. 1. What is the interest of $1 for 3yr. 9m. 18d.?

$.18	= interest of $1 for	3 years.
.045	= " " " "	9 months.
.003	= " " " "	18 days.
$.228	= " " " "	3yr. 9m. 18d., Ans.

2. What is the interest of $1 for 2yr 5m. 20d.?

$.12	= interest of $1 for	2 years.
.025	= " " " "	5 months.
.003$\frac{1}{3}$	= " " " "	20 days.
$.148$\frac{1}{3}$	= " " " "	2yr. 5m. 20d., Ans.

3. What is the interest of $1 for 5 yr. 11m. 15d.?
Ans. $.357$\frac{1}{2}$.
4. What is the interest of $1 for 1yr. 7m. 29d.?
Ans. $.099$\frac{5}{6}$.
5. What is the interest of $1 for 2yr. 4m. 4d.?
Ans. $.140$\frac{2}{3}$.
6. What is the interest of $1 for 8yr. 3m. 17d.?
7. What is the interest of $1 for 4yr. 11m. 12d.?
8. What is the interest of $1 for 14yr. 6m. 7d.?
9. What is the interest of $1 for 2yr. 1m. 5d.?
10. What is the interest of $1 for 3yr. 8m. 3d.?

195. The interest of $2 is evidently twice as much as the interest of $1; so the interest of $3, $4 or $7 is 3, 4 or 7 times the interest of $1; and the interest of $2.25 is 2.25 (i. e. 2 and 25 hundredths) times the interest of $1; $\therefore$, to find the interest of any number of dollars we have only to find the interest of $1, and then repeat that as many times as there are dollars in the principal.

11. What is the interest of \$2 for 3yr. 7m. 9d.?

$$\begin{array}{rl} \$.2\,1\,6\,\tfrac{1}{2} & = \text{interest of \$1 for 3yr. 7m. 9d.} \\ 2 & \\ \hline \$.4\,3\,3 & = \text{interest of \$2 for 3yr. 7m. 9d., Ans.} \end{array}$$

12. What is the interest of \$5.25 for 1yr. 11m. 18d.?

$$\begin{array}{rl} \$.1\,1\,8 & = \text{interest of \$1 for 1yr. 11m. 18d.} \\ 5.2\,5 & \\ \hline 5\,9\,0 & \\ 2\,3\,6 & \\ 5\,9\,0 & \\ \hline \$.6\,1\,9\,5\,0 & = \text{interest of \$5.25 for 1 yr. 11m. 18d., Ans.} \end{array}$$

13. What is the interest of \$500 for 3yr. 11m. 12d.

$$\begin{array}{rl} \$\ .2\,3\,7 & = \text{interest of \$1 for 3yr. 11m. 12d.} \\ 5\,0\,0 & \\ \hline \$1\,1\,8.5\,0\,0 & = \text{interest of \$500 for 3yr. 11m. 12d., Ans.} \end{array}$$

14. What is the interest of \$350 for 2yr. 7m. 15d.?

Ans. \$55.125.

15. What is the interest of \$15.20 for 8yr. 10m. 9d.?

16. What is the interest of \$44.44 for 4yr. 7m. 19d.?

17. What is the interest of \$18.50 for 2yr. 9m. 3d.?

18. What is the interest of \$15.33 for 3yr. 5m. 8d.?

19. What is the interest of \$12.48 for 1yr. 7m. 11d.?

20. What is the interest of \$27.57 for 3m. 10d.?

21. What is the interest of \$45.156 for 26d.?

22. What is the interest of \$47.543 for 1yr. 5m. 17d?

196. The mode of casting interest given in Art. 195 is perfectly simple, but the product is not changed when the multiplicand and multiplier change places (41). Hence, according to custom,

To cast interest, at 6 per cent. per annum, on any sum, for any time,

RULE.—*Multiply the principal by the decimal which represents the interest of* \$1 *for the given time.*

23. What is the interest of $468 for 2yr. 6m. 11d.?

FIRST OPERATION.

$$
\begin{array}{rl}
\$468. & = \text{Principal.} \\
.151\tfrac{5}{6} & = \text{int. of } \$1. \\
\hline
390 & \\
468 & \\
2340 & \\
468 & \\
\hline
\$71.058, & \text{Ans.}
\end{array}
$$

$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$. Instead of multiplying by $\frac{5}{6}$, as in this example, it is usually easier to multiply by $\frac{1}{2}$ and $\frac{1}{3}$, i. e. divide by 2 and 3, as in the following operation:—

SECOND OPERATION.

$$
\begin{array}{r}
\$468. \\
.151\tfrac{1}{2}\tfrac{1}{3} \\
\hline
234 \\
156 \\
468 \\
2340 \\
468 \\
\hline
\$71.058, \text{ Ans.}
\end{array}
$$

In like manner, when the multiplier is $\frac{2}{3}$, we may divide by 3 and set down the quotient twice.

24. What is the interest of $87.66 for 1yr. 7m. 15d.?

Ans. $8.54685.

25. What is the interest of $356 for 3yr. 8m. 18d.?

Ans. $79.388.

26. What is the interest of $965.188 for 2yr. 3m. 11d.?

Ans. $132.07—.

NOTE 1.—In the 26th example, the Ans. is $132.069891⅓, but this, in all ordinary business transactions, would be called $132.07. In the following examples in interest, only 3 decimal places *in the product* will be preserved, but if the 4th decimal place is 5 or more, the 3d place will be increased by 1 thousandth.

27. What is the interest of $356.184 for 2yr. 3m. 9d.?

$48.619, Ans.

28. What is the interest of $46.785 for 5yr. 8m. 17d.?

Ans. $16.039.

29. What is the interest of $17.49 for 1yr. 7m. 8d.?

Ans. $1.685.

30. What is the interest of $1307.87 for 2yr. 9m. 6d.?

31. What is the interest of $87.46 for 2yr. 4d.?

32. What is the interest of $.57 for 6yr. 8m.?

33. What is the interest of $1847.96 for 3yr. 6m. 3d.?

34. What is the interest of $432.50 for 8m. 27d.?

35. What is the interest of $125 from June 7, 1851, to Feb. 11, 1854? Ans. $20.083.

NOTE 2.—Ex. 35 differs from the preceding only in finding the time (170).

36. Find the interest on $76.72 from April 18, 1852, to Jan. 26, 1855. $Ans. 12.761.

37. Find the interest on $1728 from Aug. 17, 1854, to Sept. 19, 1858.

38. Find the interest on $111.111 from Oct. 12, 1853, to Dec. 30, 1857.

39. What is the interest on $37.75 from Nov. 13, 1816, to May 12, 1841?

40. What is the interest of $100 from March 26, 1818, to June 21, 1842?

197. When the principal is in pounds, shillings, pence, etc., reduce the lower denominations to the decimal of a pound (160), then proceed as with dollars and cents, and finally reduce the decimal part of the interest back to shillings, pence, etc. (161). But 3 decimal places in the multiplicand are used.

41. What is the interest of 25£ 15s. 8d. 3qr. for 2yr. 9m. 15d.? Ans. 4£ 6s. 4d. 2qr.

42. What is the interest of £144 15s. 8d. 2qr. for 1yr. 6m. 18d.? Ans. 13£ 9s. 3d. 2qr.

43. What is the interest of 75£ 6s. 4d. 1qr. from Jan. 17, 1852, to Dec. 23, 1855?

44. Find the interest on £87 10s. 9d. from July 15, 1853, to July 4, 1857.

45. What is the interest on £100 18s. 3qr. from March 4, 1853, to March 4, 1857?

198. When money is borrowed of a private individual, the interest is usually payable at the time of settlement, if that be

within a year; or, if the time is more than a year, the interest is payable annually; but, at the banks, money is usually lent for short periods of time, as, e. g., for 30, 60 or 90 days, and *the interest is paid when the money is borrowed.*

199. If the borrower promises to pay in a specified number of days, the law allows him 3 days more in which to pay; thus, if he borrows for 30 days, he need not pay until the 33d day; if for 60 days, he may pay on the 63d day; etc.

These 3 days are called "*days of grace*," and, as the borrower is not obliged to pay until the 3d day of grace, so the lender charges interest for the time specified in the note, *and* 3 *days more;* thus, if $1000 are borrowed for 60 days, the Discount Clerk will deduct $10.50, the interest for 63 days, and pay the balance, $989.50.

200. The sum deducted is called *Bank Discount*, and is the same as *Bank Interest.*

201. The interest of $1 for 60 days being 1ct., the interest of $2 is 2cts., of $3 is 3cts., of $1000 is 1000cts., etc.; but cents are reduced to dollars by dividing by 100, i. e. by moving the decimal point two places towards the left. Hence,

To calculate Bank Discount,

RULE. — *Consider the dollars in the principal, so many cents; reduce these cents to dollars by moving the separatrix two places towards the left, and the result will be the interest for* 60 *days; then. if the time is more or less than* 60 *days, modify this result as the given time may require.*

46. What is the bank discount on a note of $684.48 payable in 90 days?

$ 6.8448	= Int. for 60 days.
3.4224	= Int. for 30 days = $\frac{1}{2}$ the Int. for 60 days.
.34224	= Int. for 3 days = $\frac{1}{10}$ the Int. for 30 days.
$ 10.60944	= Int. for 93 days, Ans.

47. What is the interest of $756.48 for 30 days and grace?

$7.5648	= Int. for 60 days.
3.7824	= Int. for 30 days.
.37824	= Int. for 3 days.
$4.16064	= Int. for 33 days, Ans.

48. What is the interest of $174.25 for 45 days and grace?

$1.7425	= Int. for 60 days.
.87125	= Int. for 30 days.
.435625	= Int. for 15 days.
.087125	= Int. for 3 days.
$1.394000	= Int. for 48 days, Ans.

49. What is the interest of $469872 for 120 days and grace? $9632.376.

50. What is the interest of $764.87 for 60 days and grace?

51. What sum of money may be drawn at a bank, on a note of $468, payable in 45 days and grace? Ans. $464.256.

52. What sum may be drawn on a note of $844.28, payable in 90 days and grace?

53. What sum may be drawn on a note of $2348 for 30 days and grace?

202. When interest is required at any other *rate* than 6 per cent., *first find the interest at* 6 *per cent.; then divide this interest by* 6, *which will give the interest at* 1 *per cent.; and, finally, multiply the interest at* 1 *per cent. by the given rate.*

54. What is the interest of $346.50 for 2yr. 3m. 6d. at 5 per cent.? Ans. $39.27.

$346.50	= Principal.
.136	= Int. of $1 at 6 per cent. for 2yr. 3m. 6d.
207900	
103950	
34650	
6) $47.12400	= Int. of Principal at 6 per cent.
$7.854	= Int. of Principal at 1 per cent.
5	
$39.270	= Int. at 5 per cent., Ans.

55. What is the interest of $856.47 for 1yr. 9m. 8d. at 7 per cent.? Ans. $106.25.

56. What is the interest of $666.666 for 6yr. 6m. 6d., at 8 per cent?

57. What is the interest of $358.14 from Jan. 18, 1849, to Dec. 20, 1853, at $9\frac{1}{2}$ per cent.?

58. What is the interest of $33.45 from Aug. 19, 1853, to June 7, 1855, at 4 per cent.?

59. What is the *amount* of $500 for 3yr. 7m. 18d. at 3 per cent.? Ans. $554.50.

60. What is the amount of $48.50 for 1yr. 8m. 17d. at $4\frac{1}{2}$ per cent.?

61. What is the amount of $166.67 from Sept. 19, 1854, to Nov. 5, 1859, at 10 per cent.?

62. What is the interest of $564.18 for 3yr. 6m. 29d. at $7\frac{1}{3}$ per cent.?

63. What is the amount of $1.125 for 16yr. 8m ?

64. What is the interest of $1.125 for 33yr. 4m.?

65. What is the interest of $6666.75 for 2yr. 7m.?

66. What is the amount of $1000 from Nov. 13, 1816, to May 12, 1841, at 7 per cent.?

67. What is the interest of $1000 from May 12, 1841, to Nov. 13, 1857, at 5 per cent.?

203. To cast interest on Notes when Partial Payments have been made,

RULE.—*Find the* AMOUNT *of the principal to the time of the first payment; from this amount subtract the first payment, and the* REMAINDER *is a* NEW PRINCIPAL, *with which proceed to the time of the second payment, and so on to the time of settlement.*

EXCEPTION.—*If any payment is less than the interest due, cast the interest on the* SAME PRINCIPAL *up to the first time when the sum of the payments shall equal or exceed the interest due; then subtract the* SUM *of the payments from the* AMOUNT *of the principal.*

Ex. 1. $525.

For value received, I promise to pay John Davis, or order, five hundred and twenty-five dollars, on demand, with interest.

Daniel Trusty.

Andover, Mass., June 4, 1848.

On this note are the following indorsements: — Sept. 9, 1849, $114.20; May 15, 1850, $78.285; Aug. 6, 1851, $244.375; what was due Feb. 9, 1853? Ans. $191.003.

$525.	Principal.
39.813	Int. from June 4, '48, to Sept. 9, '49—1yr. 3m. 5d.
564.813	Amount of Principal to Sept. 9, 1849.
114.20	1st Payment.
450.613	1st Remainder, forming the 2d Principal.
18.475	Int. from Sept. 9, '49, to May 15, '50—8m. 6d.
469.088	Amount of 2d Principal to May 15, 1850.
78.285	2d Payment.
390.803	2d Remainder, forming the 3d Principal.
28.724	Int. from May 15,'50, to Aug. 6,'51—1yr. 2m. 21d.
419.527	Amount of 3d Principal to Aug. 6, 1851.
244.375	3d Payment.
175.152	3d Remainder, forming the 4th Principal.
15.851	Int. from Aug. 6, '51, to Feb. 9, '53—1yr. 6m. 3d.
$191.003	Amount due, Feb. 9, 1853, Ans.

2. $896.50. Andover, Nov. 13, 1845.

For value received, we jointly and severally promise to pay James Thrifty, or bearer, eight hundred and ninety-six dollars and fifty cents, on demand, with interest.

Jacob Principal,
John Surety.

INDORSEMENTS: — Mar. 13, 1846, $100; Dec. 25, 1846, $25; Sept. 13, 1847, $55.759; Aug. 7, 1848, $42.20; Nov. 19, 1849, $36; Dec. 1, 1850, $50.75; Jan. 16, 1851, $347.33; Apr. 22, 1852, $336; what was due Dec. 30, 1852?

$896.50	Principal.
17.93	Int. from Nov. 13, '45, to Mar. 13, '46 — 4m
914.43	Amount of Principal to March 13, 1846.
100.	1st Payment.
814.43	1st Remainder, forming the 2d Principal.
73.299	Int. from Mar. 13, '46, to Sept. 13, '47—1yr. 6m.
887.729	Amount of 2d Principal to Sept. 13, 1847
80.759	Sum of 2d and 3d Payments.
806.97	2d Remainder, forming the 3d Principal.
161.797	Int. from Sept. 13, '47, to Jan. 16, '51—3yr. 4m. 3d
968.767	Amount of 3d Principal to Jan. 16, 1851.
476.28	Sum of 4th, 5th, 6th and 7th Payments.
492.487	3d Remainder, forming the 4th Principal.
37.429	Int. from Jan. 16, '51, to Apr. 22, '52—1yr. 3m. 6d.
529.916	Amount of 4th Principal to April 22, 1852.
336.	8th Payment.
193.916	4th Remainder, forming the 5th Principal.
8.015	Int. from Apr. 22, '52, to Dec. 30, '52—8m. 8d.
$201.931	Amount due Dec. 30, 1852, Ans.

3. $1000. Andover, June 4, 1850.

For value received, we promise to pay S. Farrar, Esq., or order, one thousand dollars, on demand, with interest from date.

Higgins & Abbott.

INDORSEMENTS: — Sept. 6, 1850, $50; July 14, 1851, $150; Aug. 9, 1852, $25; May 14, 1853, $28; Oct. 15, 1853, $125; Nov. 11, 1853, $75; Nov. 13, 1854, $500; what was due Mar. 26, 1855? Ans. $282.583.

4. $756.75. Fitchburg, Mass., Jan. 12, 1852.

Four months after date, I promise to pay Thomas Eaton, or bearer, seven hundred and fifty-six dollars and seventy-five cents, with interest, for value received. Benjamin Snow, Jr.

INDORSEMENTS: — Sept. 18, 1852, $300.777; Dec. 30, 1853, $300.499; July 12, 1854, $6.566; April 24, 1855, $55.13; what was due Dec. 30, 1855? Ans. $187.38.

5. $500. Boston, Jan. 1, 1851.

For value received, I promise to pay A. B. or order, five hundred dollars, in six months, with interest afterwards. C. D.

On the above note, $350 were paid, July 1, 1852; what was due Jan. 1, 1853? Ans. $185.40.

6. $3476.875. Andover, July 18, 1849.

For value received, I promise to pay Higgins & Abbott, or order, three thousand four hundred and seventy-six dollars, eighty-seven cents and five mills, on demand, with interest.

John Flint.

INDORSEMENTS:—Oct. 6, 1849, $747.56; Jan. 15, 1850, $54.75; April 4, 1851, $47.86; Dec. 18, 1852, $995.46; April 16, 1853, $1000. What was due Aug. 4, 1854?

7. $448.50. Colchester, June 15, 1854.

For value received of Joshua Clark, I promise to pay him, or order, four hundred forty-eight and $\frac{50}{100}$ dollars, on demand, with interest at 7 per cent. Alfred B. Peirce.

INDORSEMENTS:—Dec. 6, 1854, $75; April 19, 1855, $125; Dec. 15, 1855, $10; Jan. 25, 1856, $100. What was due July 3, 1856? Ans. $183.607.

204. The rule given in Art. 203 is the one adopted by the United States Courts and most of the State Courts; but, when settlement is made within a year after interest commences, it is customary to adopt the following

RULE.—1. *Find the amount of the principal from the time when interest commenced to the time of settlement.*

2. *Find the amount of each payment from the time of payment to the time of settlement.*

3. *Subtract the sum of the amounts of the payments from the amount of the principal.*

Ex. 1. $600. Colchester, Ct., Dec. 15, 1852.

For value received, I promise to pay Hayward, Burr & Co., or order, six hundred dollars on demand, with interest.

Joshua Clark.

Indorsements:—Feb. 3, 1853, $200; June 6, 1853, $150; Aug. 9, 1853, $200. What was due Dec. 3, 1853?

$600. Principal.
34.80 Int. of Prin. from Dec. 15, '52, to Dec. 3, '53—11m. 18d.

$634.80 Amount of Principal to Dec. 3, 1853.
$200. 1st Payment.
10. Int. of 1st Payment to Dec. 3, '53–10m.

$210. Amount of 1st Payment to Dec. 3, 1853.
$150. 2d Payment.
4.425 Int. of 2d Payment to Dec. 3, '53—5m. 27d.

$154.425 Amount of 2d Payment to Dec. 3, 1853.
$200. 3d Payment.
3.80 Int. of 3d Payment to Dec. 3, '53—3m. 24d.

$203.80 Amount of 3d Payment to Dec. 3, 1853.

$568.225 Sum of the Amounts of the 3 Payments.

$ 66.575 Sum due Dec. 3, 1853, Ans.

2. $1250.75 East Dennis, Mass., June 4, 1854.

For value received, I promise to pay Christopher Sears, or order, twelve hundred fifty and $\frac{75}{100}$ dollars, on demand, with interest. Seth Crowell.

Indorsements:—Dec. 16, 1854, $300; Jan. 1, 1855, $250; Feb. 18, 1855, $400; March 15, 1855, $300. What was due May 15, 1855?

205. In every example in interest there are four elements or particulars which claim special attention, viz., Principal, Rate, Time and Interest, any three of which being given, the other can be found.

206. To find the Interest when the Principal, Rate and Time are given, has, thus far, been the object of our discussion.

The other branches of the subject give rise to the following problems:—

207. Prob. 1.—Principal, Interest and Time given, to find the Rate.

Ex. 1. At what rate per cent. must \$300 be put on interest to gain \$18 in 2 years?

\$300, at 1 per cent., will gain \$6 in 2 years; $\therefore$, to gain \$18, the rate must be the quotient of $\$18 \div \$6 = 3$. Hence,

Rule.—*Divide the given interest by the interest of the principal, for the given time, at 1 per cent.*

2. At what rate per cent. must \$370 be put on interest to gain \$55.50 in 3 years? Ans. 5.

3. If \$97 gain \$33.95 in 5 years, what is the rate per cent.?

4. If \$50 gain \$5.60 in 3yr. 6m., what is the rate per cent.? Ans. $3\frac{1}{5}$.

5. If \$75 gain \$27 in 4 years, what is the rate per cent.?

208. Prob. 2.—Principal, Interest and Rate given, to find the Time.

Ex. 6. For what time must \$200 be on interest at 6 per cent. to gain \$36?

\$200 in 1 year, at 6 per cent., will gain \$12; $\therefore$, to gain \$36, the time in years must be the quotient of $\$36 \div \$12 = 3$. Hence,

Rule.—*Divide the given interest by the interest of the principal for one year at the given rate.*

7. How long must \$133 be on interest at 7 per cent. to gain \$32.585? Ans. 3.5yr. = 3yr.6m.

8. How long must \$50 be on interest at 9 per cent. to gain \$11.85? Ans. $2.63\frac{1}{3}$yr. = 2yr. 7m. 18d.

9. How long must \$150 be on interest at 6 per cent. to *amount* to \$195? Ans. 5 years.

10. For what time must \$56 be put to interest at $8\frac{1}{2}$ per cent. to amount to \$69.09?

11. For what time must \$1000 be put to interest at 9 per cent. to gain \$495?

12. How long will it take any sum of money to double itself on interest at 6 per cent.?

13. In what time will a sum of money quadruple itself on interest at 5 per cent.?

209. Prob. 3.—Interest, Time and Rate given, to find the Principal?

Ex. 14. What principal, at 6 per cent., will gain $18 in 1yr. 6m.?

$1, in 1yr. 6m., at 6 per cent., will gain 9cts., i. e. $.09; ∴, the principal must be the quotient of $18 ÷ .09 = $200. Hence,

Rule.—*Divide the given interest by the interest of* $1 *for the given rate and time.*

15. What principal, at 5 per cent., will gain $15 in 6 months?
Ans. $600.

16. What principal, on interest at 8 per cent., will gain $100 semi-annually?

17. B endowed a professorship with a salary of $1500 per annum; what sum did he invest at 6 per cent.?

18. What sum must be invested in property yielding 8 per cent. to furnish an income of $1600 per annum?

19. What principal, at 3 per cent., will gain $74.32 in 3yr. 6m. 18d.?

20. What sum, at 6 per cent., will gain $59.76 in 1yr. 8m 27d.?

210. To the preceding we may add

Prob. 4.—Amount, Rate and Time given, to find the Principal?

Ex. 21. What principal, at 5 per cent., will amount to $110 in 2 years?

$1 in 2 years, at 5 per cent., amounts to $1.10; ∴, the principal must be the quotient of $110 ÷ 1.10 = $100. Hence,

RULE.—*Divide the given amount by the amount of* $1 *for the given rate and time.*

22. What principal, at 6 per cent., will amount to $360.585 in 16 months? Ans. $333.875.

23. What is the interest of that sum for 3 years, at 6 per cent., which will, at the given rate and time, amount to $472.
Ans. $72.

24. What sum, at 9 per cent, will amount to $3569.47 in 1yr. 8m. 12d.?

25. What is the interest of that sum for 2yr. 6m., at 8 per cent., which will, at the given rate and time, amount to $480?

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§ 18. COMPOUND INTEREST.

211. *Simple Interest* is interest on a given *principal* (191).

212. COMPOUND INTEREST is interest on both *principal and interest*, the latter not being paid when it becomes due.

213. The principal may be increased by adding the interest to it annually, semi-annually, quarterly, etc., according to agreement.

214. Compound interest, though *just*, is not *legal*.

However, if the creditor makes a legal demand for the interest when it becomes due, he may collect interest on the interest; and this is, virtually, collecting compound interest.

In like manner, a demand made for the payment of a store-bill or any other account, will enable the creditor to collect interest on the account from the time of the demand.

215. To calculate Compound Interest,

RULE.—*Make the* AMOUNT *for the* FIRST *year or specified time, the* PRINCIPAL *for the* SECOND; *the amount for the* SECOND,

the principal for the THIRD; *and so on. From the* LAST AMOUNT *subtract the* FIRST PRINCIPAL, *and the* REMAINDER *is the compound interest.*

Ex. 1. What is the compound interest on $100 for 3yr. 3mo., at 6 per cent. per annum?

$100.	1st Principal.
6.	Interest for 1 year.
106.	1st Amount or 2d Principal.
6.36	Interest of 2d Principal for 1 year.
112.36	2d Amount or 3d Principal.
6.7416	Interest of 3d Principal for 1 year.
119.1016	3d Amount or 4th Principal.
1.786524	Interest of 4th Principal for 3 months.
120.888124	4th or last Amount.
100.	1st Principal.
$ 20.888124	Compound Interest for 3yr. 3m., Ans.

2. What is the compound interest on $200 for 1yr. 8m., at 4 per cent. per annum, allowing interest to be due semi-annually?

$200.	1st Principal.
4.	Interest for 6 months.
204.	1st Amount or 2d Principal.
4.08	Interest of 2d Principal for 6 months.
208.08	2d Amount or 3d Principal.
4.1616	Interest of 3d Principal for 6 months.
212.2416	3d Amount or 4th Principal.
1.414944	Interest of 4th Principal for 2 months.
213.656544	4th or last Amount.
200.	1st Principal.
$ 13.656544	Compound Interest for 1yr. 8m., Ans.

3. What is the compound interest on $1000 for 3 years, at 7 per cent.? Ans. $225.043.

4. What is the *amount* of $5000 at compound interest, for 4yr. 10m.? Ans. $6628.004+.

5. What is the amount of $900 at compound interest for 2yr. 7m., interest payable quarterly?

6. What is the compound interest on £48 10s. for 3yr. 9m. 18d.?

7. What is the compound interest on $3476.95, at 7 per cent., for 2yr. 7m. 27d.?

8. What is the amount of 25cts. for 3yr. 5m. 16d., at 3 per cent., compound interest?

9. What is the amount of $25 for 3yr. 5m. 16d., at 3 per cent., compound interest?

10. What is the amount of $25 for 3yr. 5m. 16d., at compound interest?

11. What is the amount of $1000000 for 6yr. 6m. 6d., compound interest?

216. Compound interest may be calculated more expeditiously by means of the following

TABLE,

Showing the Amount of $1, £1., *etc., interest compounded annually at* 4, 5, 6, 7 *and* 8 *per cent., from* 1 *to* 20 *years.*

Yrs.	4 per cent	5 per cent.	6 per cent.	7 per cent.	8 per cent.	Yrs.
1	1.040000	1.050000	1.060000	1.070000	1.080000	1
2	1.081600	1.102500	1.123600	1.144900	1.166400	2
3	1.124864	1.157625	1.191016	1.225043	1.259712	3
4	1.169859—	1.215506+	1.262477—	1.310796+	1.360489—	4
5	1.216653—	1.276282—	1.338226—	1.402552—	1.469328+	5
6	1.265319+	1.340096—	1.418519+	1.500730+	1.586874+	6
7	1.315932—	1.407100+	1.503630+	1.605781+	1.713824+	7
8	1.368569+	1.477455+	1.593848+	1.718186+	1.850930+	8
9	1.423312—	1.551328+	1.689479—	1.838459+	1.999005—	9
10	1.480244+	1.628895—	1.790848—	1.967151+	2.158925—	10
11	1.539454+	1.710339+	1.898299—	2.104852—	2.331639—	11
12	1.601032+	1.795856+	2.012196+	2.252192—	2.518170+	12
13	1.665074—	1.885649+	2.132928+	2.409845+	2.719624—	13
14	1.731676+	1.979932—	2.260904—	2.578534+	2.937194—	14
15	1.800944—	2.078928+	2.396558+	2.759032—	3.172169+	15
16	1.872981+	2.182875—	2.540352—	2.952164—	3.425943—	16
17	1.947900+	2.292018+	2.692773—	3.158815+	3.700018+	17
18	2.025817—	2.406619+	2.854339+	3.379932+	3.996019+	18
19	2.106849+	2.526950+	3.025600—	3.616528—	4.315701+	19
20	2.191123+	2.653298—	3.207135+	3.869684+	4.660957+	20

Ex. 12 What is the compound interest on $600 at 6 per cent. per annum for 20yr.?

$$
\begin{array}{rl}
\$2.207135 & =\text{Int. of \$1 for 20yr. taken from the Table} \\
600 & \\
\hline
\$1324.281000 & =\text{Int. of \$600 for 20yr.---Ans.}
\end{array}
$$

13. What is the compound interest on $30 at 6 per cent. per annum for 5yr. 6m.?

$$
\begin{array}{rl}
\$1.338226 & =\text{Amount of \$1 for 5yr.} \\
.03 & =\text{Int of \$1 for 6m.} \\
\hline
.04014678 & \\
.338226 & =\text{Int. of \$1 for 5yr.} \\
\hline
\$.37837278 & =\text{Int. of \$1 for 5yr. 6m.} \\
30 & \\
\hline
\$11.35118340 & =\text{Int. of \$30 for 5yr. 6m.---Ans.}
\end{array}
$$

14. What is the amount of $50, at 7 per cent. per annum, for 15yr. at compound interest?

$$
\begin{array}{rl}
\$2.759032 & =\text{Amount of \$1 for 15yr.} \\
50 & \\
\hline
\$137.951600 & =\text{Amount of \$50 for 15yr.---Ans.}
\end{array}
$$

15. What is the amount of $350.50, at 8 per cent. compound interest, for 18yr. 7m. 12d.?

16. What is the interest of $500, at 8 per cent. per annum, compounded semi-annually for 9yr. 6m.? Ans. $553.425.

17. What is the amount of $1000, at 16 per cent. per annum, interest compounded quarterly for 3yr. 6m. 12d.?

§ 19. COMMISSION.

217. COMMISSION *is a compensation made to an agent for transacting certain kinds of business,* such e. g. as buying and selling goods, collecting and loaning money, etc.

218. This compensation is usually so much per cent. on the money collected, lent, expended, etc. Hence,

RULE. — *Multiply the sum collected, lent, expended, etc., by the rate per cent.*

Ex. 1. What shall I pay my agent for selling $1350 worth of goods, his commission being 2 per cent.? Ans. $27.

2. What shall I receive for collecting $5725, my commission being 3 per cent.?

3. The taxes in the town of A for the year 1855, were $18000. What was the cost of collecting them, $\frac{1}{5}$ of 1 per cent commission being allowed? Ans. $36.

4. My agent has lent for me $6350. His commission is $\frac{1}{4}$ of 1 per cent.; what shall I pay him?

5. An agent has purchased goods for his employer to the amount of $3484.50. What does his commission amount to, at 1 per cent.?

6. My agent has sold 34 cases of Rubber Shoes, containing 100 pairs each, at 75 cents per pair. His commission being $1\frac{1}{2}$ per cent., what shall he retain for his services and what shall he pay over to me?

Ans. His commission, $38.25; my due, $2511.75.

7. My agent in New Orleans has bought 500 cwt. 3qr. $12\frac{1}{2}$lb. of sugar at $9 per cwt.; to what does his commission of $3\frac{1}{2}$ per cent. amount?

8. Having sold 3560lb. of tea at $37\frac{1}{2}$ct., what is my commission at 2 per cent., and how much money shall I remit to my employer?

§ 20. INSURANCE.

219. INSURANCE is security against loss of property by fire, shipwreck or other casualty; or against loss of life or health by disease or disaster.

220. The sum paid for the insurance is called the *premium*, and is usually a certain per cent. on the sum insured. The per cent. varies according to the nature of the property, or the age, etc., of the person insured.

Insurance against loss by fire varies from about $\frac{3}{10}$ of 1 per cent. up to 5 per cent. or more; and some property is so hazardous that Insurance Companies decline taking the risk at *any* per centage.

221. To prevent fraudulent destruction of property, Insurance Companies will usually insure property for only about $\frac{1}{2}$ or $\frac{3}{4}$ its value, requiring the owner to risk the remainder. Property insured at one office may be insured at another by consent of the first insurers, but not so that the aggregate insured at the different offices shall exceed that portion of its value which a single office is accustomed to insure.

222. A person who gets insured in a *Mutual Insurance Company* thereby becomes a member of the company, and, to some extent, liable to pay the losses of the company. He pays the premium when he effects the insurance, and also gives his note for 5 times the premium, and this note is the basis for assessments, if the losses of the company require assessments to be made. The law also allows assessments to the extent of twice the face of this note; so that the insured is *liable* to pay 10 times the original premium in addition to that premium; yet a well conducted company is as likely to pay a dividend as to make an assessment. Besides this, the premium in a *mutual company* is usually much less than in *other* companies, being not more than from $\frac{1}{2}$ to 2 per cent. for a period of 5 years.

223. The writing or record of the contract given by the insurers to the insured is called the *policy.*

224. The insurers are frequently called the *underwriters,* especially in *marine* insurance.

225. To calculate the premium,

RULE.—*Multiply the sum insured by the rate per cent.*

Ex. 1. What is the cost of insuring $5000 on my house for 1 year, at $\frac{1}{5}$ of 1 per cent, the policy being $1.25?

Ans. $11.25.

2. What is the annual premium for insuring a manufacturing establishment in the sum of $500,000, at 2 per cent.?

3. Effected insurance on my ship, the Rover, bound to Canton, for $21500, at $3\frac{1}{4}$ per cent. What is the premium?

4. My agent at Liverpool informs me that he has shipped to me goods valued at 534£ 10s. 6d. I have effected insurance on this sum at a premium of $1\frac{3}{4}$ per cent. What must I pay, including $1.25 for the policy, the pound being worth $4.87?

Ans. $46.805.

5. What will be the annual premium for insuring $12000 for 7 years on the life of a man 25 years of age, in the Massachusetts Hospital Life Insurance Company, the premium being .97 of 1 per cent. annually? Ans. $116.40.

6. What will be the annual premium for insuring $5000 for 10 years on the life of a man 30 years of age, the premium being 1.09 per cent.?

7. What is the premium on $3000, insured on my house by the Andover Mutual Insurance Company, at $\frac{1}{2}$ per cent., for a period of 5 years? Ans. $15.

8. What is the premium for insuring 437£ 16s. 3d., at $2\frac{1}{2}$ per cent., the pound being worth $4.87?

§ 21. DISCOUNT.

226. Notes and other obligations are frequently made payable at a specified time, without interest.

DISCOUNT is an abatement or deduction made for the payment of such debt before it is due.

227. The *present worth* of a debt is, evidently, a sum which, put at legal interest, will amount to the debt at the time of its becoming due.

228. The *rule* for finding the present worth is that given in Prob. 4, Art. 210; viz.:—

Divide the given sum by the AMOUNT *of* $1 *for the given rate and time.*

229. *The* DISCOUNT *is found by subtracting the present worth from the face of the debt.*

Ex. 1. What is the present worth of a debt of $100, payable in one year, without interest?

```
1.06)$100.00($94.339+Ans.
       954
       ---
        460
        424
        ---
         360
         318
         ---
          420
          318
          ---
          1020
           954
           ---
            66
```

Present worth, discount and debt correspond to principal, interest and amount. Having the principal, we obtain the amount by multiplying the principal by the amount of $1. Conversely, dividing the amount by the amount of $1, gives the principal. *Hence the rule.*

2. What is the present worth of $756, payable in 1yr. 4m.?
Ans. $700.

3. What is the discount on $475, due in 6m. and 18d?
Ans. $15.175.

4. What is the present worth and what the discount on a note for $1800, due in 1yr. 7m.?

5. A note of $850 is due Feb. 20, 1856; what is its value on June 8, 1855?

NOTE 1.—The above rule is clearly just, but it is the almost universal custom of bankers, merchants and others, to deduct *interest* instead of *discount*. Thus, in Ex. 1, a business man would deduct $6, and of course pay $94: but $94 on interest for one year will amount to only $99 64; ∴ the lender will gain 36 cents by waiting a year and then receiving $100 instead of taking $94 to-day.

NOTE 2.—The interest on the present worth equals the discount on the debt.

6. What is the present worth of $800, due 1 year hence, at 5 per cent.?

7. What is the discount on $256, due 2yr. 3m. hence, at 8 per cent.?

8. I have a note for $600, payable 8 months from to-day; what is it worth to day?

9. I have a note for $1000, payable May 1, 1858; what is it worth to-day, June 20, 1857?

10. What shall I discount for present payment on a note of $325.75, due in 6m. 12d.?

11. May 12, 1857, gave my note to B for $40, payable in 8 months; what is the worth of this note to-day, June 24, 1857?

12. What is the interest for 6 months on the present worth of a note for $350, due 6 months hence?

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§ 22. EQUATION OF PAYMENTS.

230. EQUATION OF PAYMENTS is the method of determining when several debts due from one person to another, payable at *different* times, may be paid at *one* time, so that neither party may suffer loss.

Ex. 1. A owes B \$2, payable in 3 months, and \$5, payable in 10 months; what is the equated time for paying the two debts?

$$\begin{array}{r} 3 \times 2 = 6 \\ 10 \times \underline{5} = \underline{50} \\ 7\ \big)\,56 \\ \overline{\text{8m. Ans.}} \end{array}$$

The privilege of keeping \$2 for 3m. would be canceled by keeping \$1 6m.; so \$5 for 10m. is the same as \$1 for 50m.; ∴, for the two debts, A might keep \$1 for 56m., but as he has \$7 to keep, he may retain it only $\frac{1}{7}$ of 56m., viz. 8m. Hence, to find the equated time for paying two or more debts, due at different times,

RULE. — *Multiply each debt by the number expressing the time to elapse before it becomes due, and then divide the sum of the products by the sum of the debts.*

Ex. 2. What is the equated time for paying the following debts, viz. \$400, due in 6m.; \$500, due in 8m. and \$1000, due in 12m.? Ans. $9\frac{13}{19}$m.

3. \$360, \$548 and \$775 are due in 6, 9 and 12m. respectively; what is the equated time for payment?

4. I owe \$1000, one half payable to-day and the remainder in 6m.; when is the equated time for payment? Ans. 3m.

5. A merchant has \$600 due him in 7m., but the debtor pays him $\frac{1}{2}$ ready money and $\frac{1}{3}$ in 4m.; how long may he retain the balance? Ans. 2yr. 10m.

6. E owes F \$50 payable in 4m. and \$100 in 8m.; F owes E \$250 payable in 10m. If E makes present payment of his whole debt, how long may F retain the \$250? Ans. 6m.

7. A owes B a debt, $\frac{1}{6}$ of which is payable in 2m., $\frac{1}{3}$ in 3m. and the rest in 6m. If A makes present payment of $\frac{1}{2}$ of the debt, how long may he keep the other half?

231. Bills contracted at different times, and each having a specified credit, have their equated time of payment calculated upon precisely the same principle, by the following

RULE.— *Observe the date when each bill becomes due; also, the number of days to elapse, after the bill first payable becomes due*

to the time when each of the others is due; multiply each of these bills by its time, and divide the sum of the products by the sum of the bills; the quotient will be the number of days from the time of the earliest payment to the equated time.

Ex. 8.—Bought of Higgins & Abbott the following Bills of Goods; viz.:—

Jan. 6, 1852,	$ 40	on	90	days' credit.
Feb. 8, "	100	"	45	" "
Mar. 14, "	75	"	60	" "
Apr. 5, "	37	"	56	" "

What is the equated time for their payment?

The 2d bill (first payable)	becomes due	Mar. 24.
" 1st "	" "	Apr. 5.
" 3d "	" "	May 13.
" 4th "	" "	May 31.

From Mar. 24 to Apr. 5 = 12 days.
" " " " May 13 = 50 "
" " " " May 31 = 68 "

$$\begin{array}{rl}
\$100 & \\
40 \times 12 = & 480 \\
75 \times 50 = & 3750 \\
\underline{37} \times 68 = & \underline{2516} \\
252 \qquad) & 6746 \\
& \overline{26\tfrac{97}{126}}
\end{array}$$

∴, it will be seen by the operation in the margin, that the equated time is nearly 27 days from the 24th of Mar.; viz., Apr. 20, 1852, Ans.

9. If I buy goods of W. F. Draper, as follows; viz.:—

Jan. 16, 1853,	on	60	days' credit, a bill of	$ 75.
Mar. 9, "	"	40	" " "	50.
Mar. 15, "	"	50	" " "	100.
Apr. 1, "	"	45	" " "	25.
May 30, "	"	60	" " "	200.
June 12, "	"	50	" " "	50.

What is the equated time of payment?

10. Sold Peter Paywell the following goods; viz.: —

Jan.	1, 1856,	on	90	days'	credit,	a bill	of	$100.
Jan.	31,	"	" 75	"	"	"		75.
Feb.	12,	"	" 60	"	"	"		200.
Feb.	29,	"	" 60	"	"	"		66.
Apr.	30,	"	" 45	"	"	"		300.

At what time shall he pay me the sum of all the bills?

232. The above rules in Equation of Payments are based on the supposition that the gain to the debtor by delaying the payment of the debts first due is balanced by paying the other debts before they are due, a supposition not strictly true, as may be seen by the following: —

If I owe B $2000, $1000 of which is payable to-day and the other $1000 in 2 years, the equated time by the rule would evidently be 1 year. But if I retain the first $1000 for a year, I ought then to pay the *amount* of $1000 on interest for a year, viz., $1060, and if I pay the other $1000 at the same time, i.e. 1 year before it is due, I ought to pay its *present worth* (227), which is $\$943.39\frac{33}{53}$; ∴ if I settle at the end of 1 year, justice would require me to pay $\$1060 + \$943.39\frac{33}{53} = \$2003.39\frac{33}{53}$, which is $\$3.39\frac{33}{53}$ more than the rule demands.

The $2000 should be paid at such a time that I may, in the 2 years, gain just as much by keeping the 1st $1000 *after* it is due, as B gains by receiving the 2d $1000 *before* it is due; or, in other words, the *interest* on the 1st $1000 for the time I keep it *after* it is due, must equal the *discount* on the 2d $1000 for the time it is paid *before* it is due.

By an Algebraic solution, too complicated to be presented here, this time is found to be a little more than .97 of a year. The accuracy of this result may be verified in several ways; thus,

Interest on $2000 for .97 of a year	=	$116.40
Interest on $116.40 for 1.03 years	=	7.19+
		$123.59+, my gain.
Interest on $2000 for 1.03 years	=	$123.60, B's gain.

Or thus, Interest on \$1000 for .97 of a year = \$58.20.
Discount on \$1000 for 1.03 years = \$58.20+.

NOTE. — The results are not *exactly* equal, because the equated time is not exactly .97 of a year.

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§ 23. COMPOUND EQUATION OF PAYMENTS.

233. COMPOUND EQUATION OF PAYMENTS consists in equating the debit and also the credit side of an account, and then equating those average times.

Questions of this nature are of daily occurrence in the counting-room, and may be illustrated by the following examples:—

1. A has bought of B several bills of goods on different terms of credit, as follows:—

April 15, 1857,	on	3 months'	credit,	a bill	of	\$200.
May 1,	"	" 4	"	"	"	600.

B has also bought of A as follows:—

May 15, 1857,	on	3 months'	credit,	a bill	of	\$300.
June 14,	"	" 4	"	"	"	900.

When shall B pay to A the *balance* of his debt?

By the rule, Art. 230, A's debt of \$800 is due Aug. 20, and B's debt of \$1200, Sept. 29, 40 days after A's debt is due, and the question now is, when shall B pay the balance of \$400?

Should A and B pay their respective debts when due, all would be right; but, as it is proposed to settle by B's paying the balance of \$400, he may evidently retain the \$400 long enough after Sept. 29 to cancel the favor he has given A in allowing him to keep \$800 for 40 days after it is due; now, \$800 for 40 days equals \$400 for 80 days ($800 \times 40 = 32000$ and $32000 \div 400 = 80$), and 80 days from Sept. 29 extend to Dec. 18, the equated time, Ans.

2 C bought of D as follows:—

Jan. 1, 1857,	100 yds. cloth,	on	2m.	$300.
Jan. 15,	" 20 coats,	"	3m.	200.
Feb. 15,	" 50 shawls,	"	60 days,	500.

D also bought of C:—

Jan. 20, 1857,	10 tons hay,	on	3m.	$200.
Feb. 25,	" 100bush. corn,	"	2m.	100.
Mar. 1,	" 200bbl. apples,	"	2m.	400.

When shall C pay D the balance of $300?

The equated time for the payment of C's bills is found to be April 2, and that for D's, April 27, 25 days later. Since the larger sum is due first, it is evident the balance, $300, must be paid long enough *before* April 2 to have its interest cancel the interest on the $700 which remain unpaid for 25 days from April 2 to April 27, viz., 58 days ($700 \times 25 = 17500$ and $17500 \div 300 = 58\frac{1}{3}$), and 58 days before April 2 will carry us back to Feb. 3, the equated time, Ans.

234. From these illustrations,

To equate accounts,

RULE.—1. *Equate the debit and also the credit side of the account, and find the number of days between the equated times.*

2. *Multiply the smaller side of the account by this number of days, and divide the product by the difference between the sides; the quotient will be the number of days to be reckoned from the equated time of the larger side—to be reckoned* FORWARD *if the larger side is due at the later, and* BACKWARD *if due at the earlier date.*

235. PROOF.—*The interest on the two sides of the account from the equated time of settlement to the equated time of the sides, severally, will be alike.*

Proof of Ex. 1:—

Interest on $800 from Aug. 20 to Dec. 18, 120 days = $16.
Interest on $1200 from Sept. 29 to Dec. 18, 80 days = $16.

Proof of Ex. 2 :—

Interest on $1000 from Feb. 3 to April 2, 58 days = $9.667.

Interest on 700 from Feb. 3 to April 27, 83 days = $9.683.

The difference between the results in Ex. 2 arises from disregarding the $\frac{1}{3}$ of a day in the equated time.

NOTE. — When the larger sum is due at the earlier date, the rule may require the settlement to be made before some of the transactions have occurred, as in Ex. 2, *a result which is obviously impracticable;* but the difficulty will be removed by adding the interest on the balance due from the *equated* to the *actual time of settlement.* So also may a balance be paid before it is due, by paying the present worth of it at the time of actual settlement.

3. E bought of F as follows: —

Jan. 7, 1856,	on 1m.	credit,	a bill of	$800
Feb. 7, "	" 2m.	"	"	566.66$\frac{2}{3}$
" " "	" 3m.	"	"	433.33$\frac{1}{3}$

F also bought of E as follows: —

Jan. 18, 1856,	on 2m.	credit,	a bill of	$200
Feb. 26, "	" 4m.	"	"	1200
Mar. 1, "	" 3m.	"	"	300
" 26, "	" 3m.	"	"	800

When shall F pay E the balance? Ans. Jan. 27, 1857.

4. G owes H $800, payable June 4, 1857, and H owes G $600, payable Sept. 6, 1857; when is the equated time of settlement, and what should G pay, Sept. 6, 1857?

Ans. Aug. 26, 1856; $212.333.

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§ 24. EXAMPLES IN ANALYSIS.

236. We analyze an example when we proceed with it, step by step, according to its own conditions, without the guidance of any particular rule.

Ex. 1. If 6 barrels of flour cost \$42, what will 11 barrels cost?

Solution. — If 6bbl. cost \$42, then 1bbl. will cost $\frac{1}{6}$ of \$42, which is \$7; and if 1bbl. cost \$7, then 11bbl. will cost 11 times \$7, which is \$77, the answer.

2. If $\frac{5}{8}$ of a cask of wine cost \$35, what will 7 casks cost?

3. 20 is $\frac{5}{9}$ of what number?

4. 51 is $\frac{17}{173}$ of what number?

5. 95 is $\frac{19}{23}$ of what number?

6. If $\frac{19}{23}$ of a ton of hay cost 95 shillings, what will a ton cost?

7. If $\frac{37}{50}$ of a cask of oil is worth \$74, what is the value of 5 casks?

8. 64 is $\frac{8}{9}$ of how many times 12?

9. 72 is $\frac{9}{5}$ of how many times 4?

10. A man sold a watch for \$63, which was $\frac{9}{7}$ of its cost, what was its cost?

11. A pole is $\frac{2}{5}$ in the mud, $\frac{3}{7}$ in the water and 6 feet above water; what is the length of the pole?

12. A ship's crew have provisions sufficient to last 12 men 7 months; how long would they last 24 men?

13. A can build 35 rods of wall in 33 days, but B can build 9 rods while A builds 7; how many rods can B build in 44 days? Ans. 60.

14. $\frac{3}{7}$ of 28 is $\frac{4}{11}$ of how many fifths of 55?

15. $\frac{5}{11}$ of 44 is $\frac{4}{9}$ of how many thirds of 15?

16. $\frac{7}{3}$ of 27 is $\frac{21}{5}$ of how many twelfths of 60?

17. A fox has 39 rods the start of a hound, but the hound runs 27 rods while the fox runs 24; how many rods must the hound run to overtake the fox? Ans. 351.

18. A man being asked how many sheep he had, replied, that if he had as many more, $\frac{1}{2}$ as many more and $2\frac{1}{2}$ sheep, he should have 100; how many had he?

19. A man being asked how many sheep he had, replied, that if he had twice as many more, $\frac{1}{3}$ as many more and $3\frac{1}{3}$ sheep, he should have 70, how many had he?

20. A detachment of 2000 soldiers were supplied with bread sufficient for 12 weeks, allowing each man 14 ounces a day, but finding 105 barrels containing 200lbs. each, wholly spoiled, how many ounces may each man eat daily, that the remainder may last them 12 weeks?

21. A detachment of 2000 soldiers, having $\frac{1}{7}$ of their bread spoiled, were put upon an allowance of 12oz. each per day for 12 weeks; what was the whole weight of their bread, good and bad, and how much was spoiled?

22. A detachment of 2000 soldiers having lost 105 barrels of bread, weighing 200lbs. each, were allowed but 12oz. each per day for 12 weeks; but if none had been lost, they might have had 14oz. daily; what was the weight, including that which was lost, and how much was left to subsist on?

23. A detachment of 2000 soldiers, having lost $\frac{1}{7}$ of their bread, had each 12oz. per day for 12 weeks; what was the weight of their bread, including the part lost, and how much per day might each man have had, had none been lost?

24. A gentleman left his son an estate, $\frac{1}{4}$ of which he spent in 7 months, and $\frac{1}{6}$ of the remainder in 3 months more, when he had only $5000 remaining; what was the value of the estate?

25. The quick-step in marching being 2 paces of 28 inches each per second, what is the rate per hour? and in what time will a detachment of soldiers reach a place 60 miles distant, allowing a halt of $1\frac{1}{2}$ hours?

26. Two men and a boy were engaged to reap a field of rye; one of the men could reap it in 10 days, the other in 12, and the boy in 15 days. In how many days can the three together reap it?

27. The commander of a besieged fortress has 2lbs. bread per day for each soldier for 57 days, but, in anticipation of succor, he wishes to prolong the siege to 75 days; in that case, what must be the allowance of bread per day?

28. A merchant bought a number of bales of velvet, each containing $129\frac{17}{27}$yds., at the rate of $7 for 5yds., and sold them

out at the rate of \$11 for 7yds., and gained \$200 by the bargain; how many bales were there? Ans. 9.

29. A merchant bought a number of bales of hops, each bale containing $246\frac{18}{133}$lb., at the rate of \$3 for 11lb., and sold them at the rate of \$5 for 12lb., and gained \$248; how many bales did he buy? Ans. 7.

30. Suppose I pay $3\frac{3}{4}\frac{1}{8}$ cents per bushel for carting my wheat to mill, the miller takes $\frac{1}{16}$ for grinding, it takes $4\frac{1}{2}$ bushels of wheat to make a barrel of flour, I pay 25 cents each for barrels and $\$1\frac{1}{4}$ per barrel for carrying the flour to market, where my agent sells 60 barrels for $\$367\frac{1}{2}$, out of which he takes 25 cents per barrel for his services; what do I receive per bushel for my wheat? Ans. $87\frac{1}{2}$ cents.

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§ 25. RATIO.

237. Ratio *is the relation of one quantity to another of the same kind;* or, it is the quotient which arises from dividing one quantity by another of the same kind.

238. Ratio is usually indicated by two dots; thus, 8 : 4 expresses the ratio of 8 to 4.

The two quantities compared are the *terms* of the ratio; the first term being the *antecedent*, the second the *consequent*, and the two terms, collectively, a *couplet.*

239. Most mathematicians consider the *antecedent* a *dividend,* and the *consequent* a *divisor;*

$$\text{thus, } 8 : 4 = 8 \div 4 = \tfrac{8}{4} = 2,$$
$$\text{and } 3 : 12 = 3 \div 12 = \tfrac{3}{12} = \tfrac{1}{4}$$

but others take the *antecedent* for the *divisor,* and the *consequent* for the *dividend.*

$$\text{thus, } 8 : 4 = 4 \div 8 = \tfrac{4}{8} = \tfrac{1}{2},$$
$$\text{and } 3 : 12 = 12 \div 3 = \tfrac{12}{3} = 4.$$

Either system, rigidly adhered to, is correct; but the *first*, being considered the more simple and natural, is adopted in this work.

240. An *inverse* or *reciprocal* ratio of any two quantities is the ratio of their *reciprocals* (106); thus, the *direct* ratio of 6 to 3 is $6:3=6\div 3$, and the *reciprocal* ratio of 6 to 3 is $\frac{1}{6}:\frac{1}{3}=\frac{1}{6}\div\frac{1}{3}=\frac{1}{6}\times\frac{3}{1}=\frac{3}{6}=3\div 6=3:6$; $\therefore$ *any direct ratio* by the *first* method is a *reciprocal ratio* by the *second*, and vice versa.

241. If the antecedent *equals* the consequent, the ratio is a *unit*, and is called *a ratio of equality;* thus, $5:5=1$, is a ratio of equality.

242. If the antecedent is *greater* than the consequent, the ratio is *more than a unit*, and is called *a ratio of greater inequality;* thus, $12:4=3$, is a ratio of greater inequality.

243. If the antecedent is *less* than the consequent, the ratio is *less than a unit*, and is called *a ratio of less inequality*; thus, $2:10=\frac{1}{5}$, is a ratio of less inequality.

244. The antecedent and consequent being a dividend and divisor, it follows that any operations on them will affect the value of the ratio just as like operations on the dividend and divisor will affect the quotient; or as like operations on the numerator and denominator of a fraction will affect the value of the fraction; $\therefore$,

(a) If the antecedent be multiplied by any number, the ratio is multiplied by the same (59, a); thus,

$$12:3=4,\text{ but } 2\times 12:3=2\times 4;\text{ and}$$
$$2:6=\tfrac{1}{3},\text{ but } 2\times 2:6=2\times\tfrac{1}{3}.$$

(b) If the antecedent be divided, the ratio is divided (59, b); thus,

$$48:6=8,\text{ but } 48\div 2:6=8\div 2;\text{ and}$$
$$4:16=\tfrac{1}{4},\text{ but } 4\div 2:16=\tfrac{1}{4}\div 2.$$

(c) If the consequent be multiplied, the ratio is divided (59, c); thus,

$$30 : 5 = 6, \text{ but } 30 : 5 \times 3 = 6 \div 3; \text{ and}$$
$$3 : 12 = \tfrac{1}{4}, \text{ but } 3 : 12 \times 5 = \tfrac{1}{4} \div 5.$$

(d) If the consequent be divided, the ratio is multiplied (59, d); thus,

$$18 : 6 = 3, \text{ but } 18 : 6 \div 2 = 2 \times 3; \text{ and}$$
$$2 : 10 = \tfrac{1}{5}, \text{ but } 2 : 10 \div 2 = 2 \times \tfrac{1}{5}.$$

(e) If the terms of the ratio are both multiplied or both divided by the same number, the ratio is not changed (60, Cor. and 61, Cor.); thus,

$$12 : 3 = 4, \text{ and } 5 \times 12 : 5 \times 3 = 4; \text{ also,}$$
$$20 : 4 = 5, \text{ and } 20 \div 2 : 4 \div 2 = 5.$$

245. The ratio of two fractions that have a common denominator is the same as the ratio of their numerators; thus, $\frac{6}{20} : \frac{3}{20} = 6 : 3$, since multiplying both terms by 20 does not alter the ratio.

246. The direct ratio of two fractions that have a common numerator is the inverse ratio of their denominators; thus, $\frac{5}{6} : \frac{5}{12} = \frac{1}{6} : \frac{1}{12} = \frac{12}{72} : \frac{6}{72} = 12 : 6$; for, first, we divide the terms by 5 (244, e), then reduce them to a common denominator, and, finally multiply them by 72 (244, e).

247. The antecedent, consequent and ratio are so related to each other, that, if either *two* of them be given, the other may be found; thus, in $12 : 3 = 4$, we have

antecedent ÷ consequent = ratio,
antecedent ÷ ratio = consequent, and
consequent × ratio = antecedent.

248. When there is but one antecedent and one consequent, the ratio is said to be *simple;* thus, $15 : 5 = 3$, is a simple ratio.

249. When the corresponding terms of two or more simple ratios are multiplied together, the resulting ratio is said to be *compound;*

$$\begin{cases} 6 : 2 = 3 \\ 8 : 2 = 4 \end{cases} \qquad \begin{cases} 6 : 3 = 2 \\ 8 : 2 = 4 \\ 10 : 2 = 5 \end{cases}$$

thus, $48 : 4 = 12$ and $480 : 12 = 40$, are compound ratios.

A compound ratio is always equal to the product of the simple ratios of which it is compounded.

NOTE. — A compound ratio is not different in its *nature* from a simple ratio, but it is called *compound* merely *to denote its origin.*

(a) If each of the terms of a ratio be *squared* (94, b, Note 1), the compound ratio so formed is called a *duplicate ratio*, and is equal to the *square* of the simple ratio; thus, $6^2 : 2^2 = 3^2$, i. e. $36 : 4 = 9$, is the duplicate of $6 : 2 = 3$.

(b) If each term be *cubed* (94, b, Note 1), the result is called a *triplicate ratio*, and is equal to the *cube* of the simple ratio; thus, $4^3 : 2^3 = 2^3$, i. e. $64 : 8 = 8$, is the triplicate ratio of $4 : 2 = 2$.

(c) A similar result will be obtained by raising the terms of a ratio to *any like powers*, and also by taking *any like roots* (94, b, Note 1).

(d) If the *square* root of each term be taken, the resulting ratio is called the *sub-duplicate ratio;* if the cube root, the *sub-triplicate ratio;* etc.

(e) A *double* or *duple ratio* is twice a given ratio, and is obtained by multiplying the antecedent or by dividing the consequent by 2 (244, a and d); a *triple ratio* is three times a ratio, and is obtained by multiplying the antecedent or dividing the consequent by 3; etc. Let not the pupil confound *duple, triple, quadruple*, etc., with *duplicate, triplicate, quadruplicate*, etc.

(f) The *half, third, fourth*, etc. of a ratio are called the *sub-duple, sub-triple, sub-quadruple ratio*, etc.

What operations upon the terms of a ratio will produce the *sub-duple, sub-triple, sub-quadruple ratio*, etc.?

§ 26. PROPORTION.

250. PROPORTION *is an equality of ratios.*

251. At least 2 ratios and ∴ 4 terms are required to form a proportion.

252. The proportionality of the four numbers, 8, 4, 6 and 3, may be indicated thus,

$$8 : 4 :: 6 : 3,$$

which is read, 8 is to 4 as 6 is to 3; or, as 8 is to 4 so is 6 to 3.

Any 4 numbers are proportional, and may be written and read in like manner, if the quotient of the 1st divided by the 2d is equal to the quotient of the 3d divided by the 4th.

253. The 1st and 4th terms are called *extremes,* and the 2d and 3d, *means.* The 1st and 3d are the antecedents of the two ratios and the 2d and 4th are the consequents.

254. In a proportion the product of the extremes is equal to the product of the means; thus, in $8 : 4 :: 6 : 3$, we have $8 \times 3 = 4 \times 6$; for, from the definition of proportion, we have $\frac{8}{4} = \frac{6}{3}$, and, if each member of this equation (7, Note) be multiplied by the product of the denominators, we have $8 \times 3 = 4 \times 6$.

This is one of the easiest *tests* of proportionality.

255. Any changes in the order or magnitude of the terms of a proportion which *do not affect the* EQUALITY *of the ratios* will not destroy the proportionality. These changes are very numerous; some of them will be noticed in the Supplement.

256. Since the product of the extremes is equal to the product of the means, any one term may be found when the other three are given; for the product of the extremes divided by either mean will give the other mean, and the product of the means divided by either extreme will give the other extreme.

257. When *three* numbers are in proportion, as, e. g., 4 : 6 : : 6 : 9, the 2d is called a *mean proportional* between the 1st and 3d, and the 3d, a *third proportional* to the 1st and 2d.

(a) A mean proportional between two numbers may be found by multiplying the two given numbers together and then resolving the product into *two equal factors;* thus, the mean proportional to 2 and 8 is 4, for $2 \times 8 = 16 = 4 \times 4$; $\therefore 2 : 4 :: 4 : 8$.

(b) A third proportional to two numbers may be found by *dividing the square of the 2d by the 1st.* The third proportional to 5 and 10 is 20; for $10^2 \div 5 = 20$; $\therefore$ 5 : 10 :: 10 : 20.

258. In all examples in *Simple Proportion* there are three numbers given to find a fourth; $\therefore$ Proportion is often called the *Rule of Three.*

Two of the three given numbers must be of the same kind, and the other is of the same kind as the answer.

Ex. 1. If 3 men build 6 rods of wall in a day, how many rods will 5 men build?

This example may be analyzed as follows:—If 3 men build 6 rods, 1 man will build $\frac{1}{3}$ of 6 rods, i. e. 2 rods; and if one man build 2 rods, 5 men will build 5 times 2 rods, i. e. 10 rods, Ans.; but, to solve it by proportion, we say that the given number of rods has the same ratio to the required number of rods that 3 men have to 5 men: thus,

3 men : 5 men : : 6 rods : required number of rods.

Now, since the means and 1st extreme are given, we find the 2d extreme by dividing the product of the means by the given extreme (256); thus,

$$6 \times 5 = 30 \text{ and } 30 \div 3 = 10 \text{ Ans. as before.}$$

Every example in Simple Proportion is solved in like manner. Hence,

RULE.—*Write that given number which is of the same kind as the required answer, for the third term; consider whether the*

nature of the question requires the answer to be greater or less than the third term; if greater, write the greater of the two remaining numbers for the second term, and the less for the first; but, if less, write the less for the second and the greater for the first; in either case, divide the product of the second and third terms by the first, and the quotient will be the term sought.

NOTE.—If the first and second terms are in different denominations, they should be reduced to the same before stating the question.

REMARK.—Every one who intelligently solves an example by proportion, does, in effect, solve it by analysis, but the Teacher should use much care on this point, since the scholar learns much faster when he analyzes a question than when he merely follows a rule.

Let the following examples be solved by analysis and by proportion.

Ex. 2. If a man earn \$24 in 2 months how much will he earn in 9 months?

$$\begin{array}{r} 2 : 9 :: 24 : \text{4th term.} \\ 9 \\ \hline 2\,)\,216 \\ \hline \$108, \text{ Ans.} \end{array}$$

Since we are seeking for dollars, we make \$24 the 3d term, and then, as a man will earn more in 9 months than he will in 2 months, we make 9 the 2d term and 2 the 1st. To analyze the above we say,—If a man earn \$24 in 2 months, then in 1 month he will earn $\frac{1}{2}$ of \$24, i. e. \$12; and if he earn \$12 in 1 month, then in 9 months he will earn 9 times \$12, i. e. \$108, Ans.

3. If a staff 3 feet long casts a shadow 4 feet, what is the hight of a steeple which, at the same time, casts a shadow 240 feet? Ans. 180 feet.

4. If a staff 3 feet long casts a shadow 4 feet, how long is the shadow of a steeple which is 180 feet high, at the same time?

5. If a steeple 180 feet high casts a shadow 240 feet, what is the hight of a staff which, at the same time, casts a shadow 4 feet?

6. If a steeple 180 feet high casts a shadow 240 feet, what is the length of the shadow cast by a staff 3 feet high, at the same time?

7. If a locomotive run 39000 miles in 13 weeks, how far, at that rate, would it run in 52 weeks?

BY PROPORTION.

13 : 52 : : 39000 : 4th term.

```
            52
         -----
         78000
        195000
       -------
   13) 2028000 (156000, Ans.
       13
       --
        72
        65
        --
         78
         78
         ---
          000
```

BY CANCELING.

$$\frac{39000 \times \overset{4}{\cancel{52}}}{\cancel{13}} = 156000, \text{ Ans.}$$

or,

$$\frac{\overset{3000}{\cancel{39000}} \times 52}{\cancel{13}} = 156000, \text{ Ans.}$$

8. If 33 men perform a piece of work in 67 days, in how many days will 11 men perform the same? Ans. 201.

9. If a man's salary amounts to $2700 in 3 years, what will it amount to in 11 years?

10. If a man's salary amounts to $9900 in 11 years, in how many years will it amount to $2700?

11. If I pay 2s. 8d. per week for pasturing 2 cows, what shall I pay for pasturing 11 cows?

```
2 : 11 : : 2s. 8d. :
              11
        -----------
        2) 29s. 4d.
        -----------
    Ans. 14s. 8d.
```

or,

```
2 : 11 : : 32d. :
            11
        --------
        2) 352d.
        --------
  Ans. 176d. = 14s. 8d.
```

12. If I pay 2s. 8d. for pasturing 2 cows, how many cows can be pastured the same time for 14s. 8d.?

BY PROPORTION.

$$\begin{array}{r} 32\text{d.} : 176\text{d.} :: 2 \\ \underline{2} \\ 32)\ 352\ (11, \text{ Ans.} \\ \underline{32} \\ 32 \\ \underline{32} \end{array}$$

BY CANCELING.

$$\frac{\overset{11}{\cancel{2} \times \cancel{176}}}{\underset{\cancel{16}}{\cancel{32}}} = 11, \text{ Ans.}$$

13. If 13a. 2r. 6rd. of land is worth 130£. 9s. 6d., what is the value of 94a. 3r. 2rd.?

STATEMENTS BY PROPORTION.

13a. 2r. 6rd. : 94a. 3r. 2rd. : : 130£. 9s. 6d. : 4th term;
or, 13.5375a. : 94.7625a. : : 130.475£. : 4th term;
or, 2166rd. : 15162rd. : : 31314d. : 4th term.

STATEMENT BY CANCELING.

$$\frac{130.475 \times \overset{7}{\cancel{94.7625}}}{\cancel{13.5375}} = 913.325£ = 913£.\ 6s.\ 6d., \text{ Ans.}$$

STATEMENT BY CANCELING.

$$\frac{31314 \times \overset{7}{\cancel{15162}}}{\cancel{2166}} = 219198d. = 913£.\ 6s.\ 6d., \text{ Ans.}$$

14. If 94a. 3r. 2rd. of land cost 913£. 6s. 6d., how much land may be bought for 130£. 9s. 6d.?

15. If 13a. 2r. 6rd. of land cost 130£. 9s. 6d., how much may be bought for 913£. 6s. 6d.?

16. If 94a. 3r. 2rd. of land cost 913£. 6s. 6d., what will 13a. 2r. 6rd. cost?

17. If $12\frac{1}{2}$ yds. of silk that is $\frac{3}{4}$ yd. wide will make a dress, how many yds. of muslin that is $1\frac{3}{8}$ yd. wide will be required to line it? Ans. $6\frac{9}{11}$.

18. If $\frac{7}{16}$ of a ship cost $1163, what is $\frac{9}{16}$ of her worth?

19. If 6 men perform a piece of work in 40 days, how long will it take 10 men to do the same?

20. If $\frac{3}{7}$ of a barrel of flour cost $3.60, what will 14 barrels cost?

21. If $\frac{4}{9}$ of an acre of land is worth $36.40, what is the value of 15 $\frac{3}{8}$ acres at the same price?

22. If 6 men can mow 12a. 3r. 16rd. of grass in 2 days, by working 6 hours per day, how many days will it take them if they work only 4 hours per day?

23. If 2bbl. of flour are worth as much as 3 cords of wood, how many barrels of flour will pay for 45 cords of wood?

24. If 4 men can perform a piece of work in 16 days, how many men must be added to the number to perform the same in 4 days?

25. A ship's crew of 15 men is provisioned for 30 days; how many men must be discharged that the provision may last 15 days longer?

26. A bankrupt, owing $25000, has property worth $15000; how much will be received on a debt of $500?

27. A man, owning $\frac{3}{5}$ of a ship, sells $\frac{2}{3}$ of his share for $20000; what is the value of the ship?

28. Borrowed $300 for 9 months; for how long must $450 be lent to repay the favor?

29. If, when flour is worth $12 per barrel, a penny loaf weighs 4oz., what ought it to weigh when flour is worth $8 per barrel?

30. A and B hired a pasture for $45.90, in which A pastured 11 oxen and B 19; what shall each pay?

31. If 13 men perform a piece of work in 45 days, how many men must be added to perform the same in $\frac{1}{9}$ of the time?

32. How many yards of cloth $\frac{3}{4}$ of a yard wide are equal to 63 yards $1\frac{1}{4}$ yards wide?

33. If 10 horses eat 35 bushels of oats in 2 weeks, how many bushels will 14 horses eat in the same time?

34. If the interest on $700 is $42 in one year, what will be the interest on the same sum for $3\frac{1}{2}$ years?

35. How many yards of paper 2 feet in width will paper a room that is $13\frac{1}{2}$ feet long, 12 feet wide and 9 feet high?

36. If I pay $168 for 63 gallons of wine, how much water shall I add that I may sell it at $2 per gallon without loss?

37. A certain house was built by 30 workmen in 97 days, but, being burned, it is required to rebuild it in 30 days; how many men must be employed?

38. A garrison of 1500 men has provisions for 12 months; how long will the same provisions last if the garrison is re-enforced by 300 men?

39. If a piece of land 20 rods long and 8 rods wide contain an acre, how long must it be to contain the same when it is but 2 rods wide?

40. If the earth revolves 366 times in 365 days, in what time does it revolve once? Ans. 23h. $56\frac{4}{61}$m.

41. A wall which was to be built 24 feet high was raised 8 feet by 6 men in 12 days; how many men must be employed to complete the wall in 12 days more?

42. A wall was completed by 12 men in 12 days; how many men would complete the same in 4 days?

43. Paid $3.50 for 7lb. of tea; what should I pay for 19lb.?

44. If a man perform a journey in 6 days when the days are 12 hours long, in how many days of 8 hours each will he perform the same?

45. Lent a friend $500 for 3 months; afterwards he lent me $300. How long may I retain it to balance the favor?

46. If 9 yards of muslin that is $1\frac{1}{2}$ yards wide will make a dress, how many yards of lining will be required, that is but $\frac{7}{8}$ of a yard wide?

47. A cistern has a pipe that will fill it in 6 hours; how many pipes of the same size will fill it in 45 minutes?

48. A cistern has 3 pipes; the first will fill it in 3 hours, the second in 4 hours and the third in 5 hours. In what time will they together fill the cistern? Ans. $1\frac{13}{47}$ hours.

49. A can cut a field of grain in 8 days; A and B can cut it in 6 days. In what time can B do the same?

50. If 2 horses can draw a load of 16 tons upon a railway, how many horses will be required to draw 72 tons?

51. A farm was sold at \$25.50 per acre, amounting to \$1925.25; how many acres did the farm contain?

Ans. 75a. 2r.

52. Bought a horse for \$75, and a pair of oxen for a price which was to the price of the horse in the duplicate ratio of 7 to 5; what was the price of the oxen?

53. A garrison of 300 men has provisions to last 60 days; how long will the same provisions last if the garrison is re-enforced by 100 men?

54. A garrison of 1000 men have 14oz. of bread each per day for 120 days; how long will the same bread last them if each man is allowed but 12oz. per day?

55. If $\frac{5}{11}$ of a ship cost \$25000, what is $\frac{1}{16}$ of her worth?

56. At \$27 per cwt., what is the cost of $37\frac{1}{2}$lb.?

57. If .25 of a piece of land are worth \$750, what are .376 of it worth?

58. A's property is to B's in the triplicate ratio of 3 to 4; B's estate is worth \$12800; what is the value of A's.?

59. The earth moves 19 miles per second in her orbit; how far does she go in 3m. 27sec.?

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§ 27. COMPOUND PROPORTION.

259. COMPOUND PROPORTION is an *equality* of two ratios one of which is *compound* and the other *simple*; thus,

$$\left.\begin{matrix} 3 : 12 \\ 16 : \ 2 \end{matrix}\right\} :: 18 : 9, \text{ is a compound proportion;}$$

and 48 : 24 :: 18 : 9, is the same reduced to a simple form.

NOTE.—The *compound ratio* may consist of any number of couplets.

260. Every compound proportion may be reduced to a simple form, and, moreover, every example in compound proportion may be solved by means of two or more simple proportions.

Ex. 1. If 6 men in 8 hours thresh 30 bushels of wheat, in how many hours will 2 men thresh 5 bushels?

BY SIMPLE PROPORTION.

$$2 : 6 :: 8 : 24, \text{ and}$$
$$30 : 5 :: 24 : 4, \text{ Ans.}$$

In solving this question by simple proportion, we, in the first place, disregard the *amount of labor*, and inquire how long it will take 2 men to do as much as 6 men in 8 hours. Having found 24 hours to be the answer to this question, we next disregard the *number of men*, and inquire how long it will take to thresh 5 bushels of wheat if 30 bushels are threshed in 24 hours, and thus obtain 4 hours, the true answer to the question.

BY COMPOUND PROPORTION.

$$\left.\begin{array}{r} 2 : 6 \\ 30 : 5 \end{array}\right\} :: 8 : 4, \text{ Ans.}$$

To shorten the work, we may consider both conditions at once. It will be seen that, of the first two couplets, $\left\{\begin{array}{r} 2 : 6 \\ 30 : 5 \end{array}\right\}$, one is a ratio of *less* and the other of *greater inequality* (243 and 242); but there is no impropriety in this, for one condition of the question requires the answer to be greater than the 3d term, and the other condition requires it to be less.

261. There is no *new principle* in Compound Proportion Hence,

To solve questions in Compound Proportion,

RULE.—*Write that given number which is of the same kind as the required answer for the 3d term; take any two of the remaining terms* THAT ARE ALIKE, *and, considering the question as* DEPENDING ON THESE ALONE, *arrange them as in simple proportion; arrange each pair of* LIKE TERMS *by the same principles; and then multiply the continued product of the 2d terms by the 3d term, and divide this result by the continued product of the 1st terms; the quotient will be the term sought.*

NOTE.—The work may frequently be much abridged by canceling any factor in the 2d and 3d terms, with a like factor in the 1st terms (127, a, Note 2).

Ex. 2. If 6 men in 15 days earn \$135, how many dollars will 9 men earn in 18 days?

$$\left.\begin{array}{l} 6 \text{ men} : 9 \text{ men} \\ 15 \text{ days} : 18 \text{ days} \end{array}\right\} :: \$135 : \$\text{—}?$$

$9 \times 18 \times 135 = 21870 =$ continued product of 2d and 3d terms.
$6 \times 15 = 90 =$ continued product of 1st terms.
$21870 \div 90 = 243$, Ans.

THE SAME CANCELED.

$$\left.\begin{array}{l} \left.\begin{array}{r|r} \cancel{6} & : 9 \\ \cancel{15} & : \cancel{18} \\ \cancel{5} & \cancel{3} \end{array}\right\} :: \overset{}{\underset{27}{\cancel{135}}} : \text{—}? \\ 9 \times 27 = 243, \text{ Ans.} \end{array}\right\} \text{ or, } \left\{ \frac{9 \times \overset{\cancel{3}}{\cancel{18}} \times \overset{27}{\cancel{135}}}{\cancel{6} \times \underset{\cancel{5}}{\cancel{15}}} = 243, \text{ Ans.} \right.$$

3. If 3 men, in 16 days of 12 hours each, build a wall 30ft. long, 8ft. high and 3ft. thick, in how many days of 9 hours each can 9 men build a wall 45ft. long, 9ft. high and 6ft. thick?

Ans. 24.

$$\left.\begin{array}{rl} 9 \text{ men} & : 3 \text{ men} \\ 9 \text{ hours} & : 12 \text{ hours} \\ 30\text{ft. long} & : 45\text{ft. long} \\ 8\text{ft. high} & : 9\text{ft. high} \\ 3\text{ft. thick} & : 6\text{ft. thick} \end{array}\right\} :: 16 \text{ days} : \text{—}? \text{ days.}$$

4. A wall, which was to be built 32 feet high, was raised 8 feet by 6 men in 12 days; how many men must be employed to finish the wall in 6 days? Ans. 36.

5. If 3 men, in 16 days of 12 hours each, build a wall 30ft. long, 8ft. high and 3ft. thick, how many men will be required to build a wall 45ft. long, 9ft. high and 6ft. thick, in 24 days of 9 hours each?

6. If a family of 6 persons spend \$600 in 8 months, how many dollars will be required for a family of 10 persons in 14 months? Ans. \$1750.

7. If the transportation of 9hhds. of sugar, each weighing 12 cwt., 20 leagues, cost \$50, what must be paid for the transportation of 50 tierces, each weighing $2\frac{1}{2}$cwt., 300 miles?

8. If \$100 gain \$8 in 1 year, what will \$300 gain in 9 months?

9. If \$300 gain \$18 in 9 months, what will \$100 gain in 1 year?

10. If \$100 gain \$8 in 1 year, in what time will \$300 gain \$18?

11. If \$100 gain \$8 in 1 year, what principal will gain \$18 in 9 months?

12. If \$300 gain \$18 in 9 months, what is the rate per cent.?

13. If a 2 penny loaf weighs 8oz. when wheat is 6s. 9d. per bushel, how much bread may be bought for 3s. 4d. when wheat is worth 13s. 6d. per bushel? Ans. 5lbs.

14. If a bar of silver 2ft. 1 inch long, 6in. wide and 3in. thick, be worth \$2725, what is the value of a bar of gold 1ft. $9\frac{1}{16}$in. long, 8in. wide and 4in. thick, the specific gravity of silver to that of gold being as 10.47 to 19.26, and the value per oz. of silver being to that of gold as 2 to 33? Ans. \$128293.

15. If 496 men, in 5 days of 12h. 6m. each, dig a trench of 9 degrees of hardness 465 feet long, $3\frac{2}{3}$ feet wide and $4\frac{2}{3}$ feet deep, how many men will be required to dig a trench of 2 degrees of hardness $168\frac{3}{4}$ feet long, $7\frac{1}{2}$ feet wide and $2\frac{4}{5}$ deep, in 22 days of 9 hours each? Ans. 15.

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§ 28. CONJOINED PROPORTION.

262. CONJOINED PROPORTION (frequently called the *Chain Rule* and also *Arbitration of Exchange*) is a species of Compound Proportion, in which the antecedent and consequent of each couplet are in different denominations, but equivalent in value, and each antecedent is in the same denomination as the consequent in the following couplet.

263. The rule is principally employed in the operations of exchange in the currencies of different countries; but, to unfold its principles, we will apply it to one or two simple examples in reduction.

Ex. 1. If 4qr. = 1 d., 12d. = 1s. and 20s. = 1£, how many farthings are equal to 3£?

3£ = how many qr.?
If 20s. = 1£,
12d. = 1s.
and 4qr. = 1d.

Ans. 2880qr. = 3£.

Here, evidently, the continued product of the numbers on the left of the signs of equality, will be the answer; but, had the question read thus:— If 4qr. = 1d., 24d. = 2s. and 20s. = 1£, how many farthings are equal to 3£?, we would have arranged the numbers as in the margin, and then, evidently, the continued product of the numbers on the left of the signs of equality divided by 2, (i. e. by the continued product of the numbers on the right of the signs of equality,) will be the number of farthings in 3£, as before.

3£ = how many qr.?
If 20s. = 1£,
24d. = 2s.
and 4qr. = 1d.

5760qr. = 2 × 3£,
and 5760qr. ÷ 2 = 2880qr. = 3£.

(a) This principle is equally applicable to examples in reduction ascending.

Ex. 2. If 1£ = 20s., 1s = 12d and 1d. = 4qr., how many pounds are equal to 14400qr.?

14400qr. = how many £?
If 1d. = 4qr.,
1s. = 12d.
and 1£ = 20s.

14400 ÷ 960 = 15£, Ans.

Here, as in the preceding example, the continued product of the left hand members of the equations, divided by the continued product of the right hand members, gives the correct result.

This, and all other examples, may also be modified in the same manner as example 1; thus.— If 1£ = 20s., 3s. = 36d. and 5d. = 20qr., how many pounds are equal to 14400qr.?

$$\begin{array}{rl} 14400\text{qr.} = & \text{how many £?} \\ \text{If } 5\text{d.} = & 20\text{qr.,} \\ 3\text{s.} = & 36\text{d.} \\ \text{and } 1£ = & 20\text{s.} \\ \hline 216000 \div & 14400 = 15£, \text{ Ans.} \end{array}$$

It is an *axiom* that, if equal quantities are multiplied by equals, the products will be equal. Now, in Conjoined Proportion, we have a series of equations, in which all the members are given, except the 2d member of the 1st equation; ∴ if the continued product of the 1st members be divided by the continued product of the *known* 2d members, the quotient *must necessarily* be the *unknown* 2d member, *which is the number sought.*

Ex. 3. If 5 gal. molasses are equal in value to 2 bush. corn, and 18 bush. corn to 3 cords of wood, how many gallons of molasses are equal in value to 7 cords of wood?

$$\begin{array}{lrl} & 7\text{cords} = & \text{——gal.?} \\ \text{If} & 18\text{bush.} = & 3\text{cords} \\ \text{and} & 5\text{gal.} = & 2\text{bush.} \\ \hline & 630 \div & 6 = 105\text{gal., Ans.} \end{array}$$

In Ex. 1, 3£ may be called the *demanding term;* in Ex. 2 and 3, 14400qr. and 7cords are the demanding terms.

All similar examples may be solved in like manner. Hence,

To solve questions in Conjoined Proportion,

RULE. — *Write the demanding term, and at the right of it a blank in the place of the term sought, with the sign = between them; write the term of the same name as the demanding term under the blank, and that which is equal to it in value at the left of it, with the sign = between them; and thus proceed, writing the term of the same name as the one last placed, on the right and the term equal to this on the left, till all the terms are written; then the continued product of the left-hand members, divided by the continued product of the right-hand members, will be the term sought.*

NOTE. — We may cancel here as elsewhere.

Ex. 4. If 9£ sterling money equals 12£ N. E. currency, 6£ N. E. equals 8£ N. Y., 16£ N. Y. equals 15£ N. J. and 45£ N. J. equals 28£ Ga. currency, how many pounds sterling are equal in value to 56£ Ga. currency? Ans. 54

5. If 20 acres of land in Andover are worth as much as 30 acres in Boxford, 50 in Boxford as much as 45 in Methuen, 25 in Methuen as 20 in Lawrence, and 33 in Lawrence as 40 in Bradford, how many in Andover are equal in value to 1800 in Bradford?

6. If 20 boys will perform as much labor as 15 men, and 12 men as much as 18 women, how many boys would be required to perform as much labor as 27 women?

7. If 483 American eagles are equal in value to 1000 English sovereigns, 150 English sovereigns to 161 French Louis d'ors, 22 Louis d'ors to 45 Italian sequins, 95 sequins to 44 German Carolins, and 88 Carolins to 190 Swiss ducats, how many American eagles are equal in value to 50 Swiss ducats? Ans. 11.

264. All these examples may be solved by a series of simple proportions, and, of course, by compound proportion. Each example may also be analyzed in various different ways. For illustration, let us take Ex. 3: — If 5gal. molasses are equal in value to 2bush. corn, and 18bush. corn to 3cords of wood, how many gallons of molasses are equal in value to 7cords of wood?

BY SIMPLE PROPORTION.

3cords : 7cords :: 18bush. : 42bush., and
2bush. : 42bush. :: 5gal. : 105gal., Ans

BY COMPOUND PROPORTION.

$$\left.\begin{array}{l} 3 : 7 \\ 2 : 18 \end{array}\right\} :: 5 : 105, \text{ Ans.}$$

BY ANALYSIS.

Since 3cords = 18bush.,
1cord = 6bush.;
and since 5gal. = 2bush.,
15gal. = 6bush.;
∴ 1cord = 15gal.,
and 7cords = 105gal., Ans.

§ 29. PROFIT AND LOSS.

265. "PROFIT AND LOSS," as a commercial term, signifies the gain *or* loss in business transactions. The rule may refer to the *absolute* gain or loss, or to the *percentage* of gain or loss on the *purchase price* of the property considered.

266. PROB. 1.—To find the absolute gain or loss on a quantity of goods sold at retail, the purchase price of the whole quantity being given,

RULE.—*Ascertain the whole sum received for the goods, and the difference between this and the purchase price will be the gain or loss.*

Ex. 1. Bought 16bbl. of flour for $100 and sold it at $7 per bbl.; did I gain or lose? how much total and per bbl.?

Ans. Gained $12 total; 75c. per bbl.

2. Bought 75yds. broadcloth for $250 and sold it at $4 per yd.; did I gain or lose? how much total and per yd.? Ans.

3. Bought 13cwt. 1qr. 19lb. of sugar for $107.52 and sold it at $6\frac{1}{4}$cts. per lb.; did I gain or lose? how much total and per lb.?

Ans. Lost $23.52 total; $1\frac{3}{4}$c. per lb.

4. Bought 164 yards of broadcloth and $\frac{7}{4}$ as many yards of cassimere for $1107; sold the broadcloth at $3 per yd. and the cassimere at $\frac{3}{4}$ as much per yd. Did I make or lose? how much?

267. PROB. 2.—To find the percentage of gain or loss.

RULE 1.—*Find the total gain or loss by Art.* 266, *and then say, as the purchase price is to the total gain or loss, so is* 100 *per cent. to the gain or loss per cent.*

NOTE.—The *par value* of an article is its first cost.

Ex. 1. What do I gain per cent. if I buy flour at $6 and sell it at $7 per bbl.

Cost	:	Gain	: :	Par Value	:	Gain per cent.
$6	:	$1	: :	100 per cent.	:	$16\frac{2}{3}$ per cent., Ans.

2. Bought a share of the Boston & Maine Railroad stock for \$108 and sold it for \$105? what was my loss per cent.?

$$\$108 : \$3 :: 100 : 2\tfrac{7}{9}, \text{ Ans.}$$

(a) RULE 2.—*Make a common fraction, writing the gain or loss for the numerator and the cost of the article for the denominator, and then reduce the fraction to a decimal.*

Thus, in Ex. 2, \$3 is the loss and \$108 the cost, ∴ the loss is $\frac{3}{108}$ of the purchase money and $\frac{3}{108} = .02\frac{7}{9}$; i. e. $2\frac{7}{9}$ hundredths or $2\frac{7}{9}$ per cent. of the cost.

$\frac{3}{108} = .02\frac{7}{9}$, Ans.

3. Purchased a quantity of merchandise for \$3496 and sold the same for \$3670.80; what did I gain per cent.?

Ans. 5 per cent.

4. Bought a quantity of goods for \$3496 and sold the same to John Smith on his note at 60 days for $\$3709.75\frac{475}{1979}$. This note was, on the same day, discounted at the Andover Bank. Did I make or lose? how much per cent.?

Ans. Gained 5 per cent.

5. Bought a flock of sheep at \$4 per head and sold them at \$5, what per cent. was gained?

6. Bought sheep at \$5 per head and sold them at \$4; what per cent. was lost?

7. Bought goods for \$4000, and, in one year, sold the same for \$4310, out of which paid \$190 for storage, etc.; how much per cent. on the first cost was lost?

268. PROB. 3. — The purchase price of an article being given to find such a selling price as to gain or lose a certain per cent. on the cost,

RULE 1.—*Multiply the purchase price by the per cent. to be gained or lost, written decimally, and add the product to or subtract it from the purchase price.*

Ex. 1. Bought goods for $400; how must the same be sold so as to gain 25 per cent.?

```
  $400
    .25
  ----
  2000
  800
 -----
$100.00 = gain.
$400.
------
$500. Ans.
```

This is the same as finding the *amount* of a sum of money on interest for 1 year at 25 per cent.

2. Bought a horse for $150, but it being injured, I am willing to lose 10 per cent.; for what may I sell him?

```
$150
  .10
-----
$15.00 = loss.
$150 — $15 = $135, Ans.
```

This is the same as finding the present worth of a sum due a year hence, discounting as at the banks.

(a) RULE 2. — *As* 100 *is to* 100 *increased by the per cent. to be gained or diminished by the per cent. to be lost, so is the pur chase price to the selling price.*

By this rule the 1st example will be solved thus: —

Par value : Premium value :: 1st Cost : Selling price.
100 per cent. : 125 per cent. :: $400 : $500, Ans.

The 2d example is solved thus: —

100 : 90 :: $150 : $135, Ans.

Ex. 3. Bought pepper at $12\frac{1}{2}$ cts. per lb.; how shall it be sold to lose 10 per cent.? Ans. 11c and $2\frac{1}{2}$m.

4. Bought 3 cwt. of sugar at $12\frac{1}{2}$ cts.; how shall the same be sold per lb. so as to gain 20 per cent.?

5. Bought 6 shares of Exchange Bank Stock at $100 per share; how shall the same be sold to gain $6\frac{1}{2}$ per cent.?

269. PROB. 4. — To find the first cost of an article when we know the selling price and the gain or loss per cent. on the cost,

RULE. — *Say as* 100 *increased by the per cent. to be gained or*

diminished by the per cent. to be lost is to 100, *so is the selling price to the purchase price.*

Ex. 1. Sold wheat at \$1.50 per bushel, and thereby gained 25 per cent. on the cost; what was the purchase price?

Premium value	: Par value	:: Selling price	: Cost
125 per cent.	: 100 per cent.	:: \$1.50	: \$1.20, Ans.

2. Sold apples at \$1.75 per bbl. and thereby lost 10 per cent. on the cost; what was the cost?

$$90 : 100 :: 1.75 : \$1.94\tfrac{4}{9}, \text{ Ans.}$$

3. Sold 6 yds. cloth for \$30 and gained 12 per cent. on the cost; what was the purchase price per yard?

4. Sold 10 shares of the Fitchburg R. R. stock for \$800, and thereby lost 20 per cent. on the cost; what did I pay per share?

270. PROB. 5.—If goods be sold at a certain price, and there be gained or lost a certain per cent. on the cost, to find what would be gained or lost per cent., if sold at some other price,

RULE.—*As the actual price is to the proposed price, so is* 100 *increased or diminished by the gain or loss per cent. to* 100 *increased or diminished by the gain or loss per cent. when sold at the proposed price.*

Ex. 1. Sold flour at \$7 per bbl. and thereby gained $16\frac{2}{3}$ per cent.; what per cent. should I have gained if I had sold it at \$7.50?

Actual price	: Proposed price	:: 100 + gain per cent.	: 100 + gain per cent.
\$7	: \$7.50	:: $116\frac{2}{3}$	: 125, and

125 per cent. — 100 per cent. = 25 per cent., Ans.

2. Sold beef at \$6 per cwt., and thereby lost 5 per cent.; should I have gained or lost, and how much per cent., had I sold it at \$7?

$\$6 : \$7 :: 95 : 110\frac{5}{6}$, and $110\frac{5}{6} - 100 = 10\frac{5}{6}$ per cent. gain, Ans.

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3. Sold a watch for \$21 and gained 5 per cent. on the cost; had I sold it for \$18 should I have gained or lost, and how much per cent.?

$$\$21 : \$18 :: 105 \text{ per cent.} : 90 \text{ per cent.}$$
$$100 - 90 = 10 \text{ per cent. loss, Ans.}$$

4. Sold a farm for \$5000 and thereby made 25 per cent.; should I have gained or lost, and how much per cent., if I had sold it for \$3000?

5. Sold a pair of oxen for \$175 and gained 5 per cent.; what per cent. should I have gained if I had sold them for \$200?

6. Sold a house for \$4000 and gained 20 per cent.; should I have gained or lost, and how much per cent., if I had sold it for \$3000?

7. Sold a house for \$5000 and gained 25 per cent.; what per cent. should I have gained if I had sold it for \$6000?

271. PROB. 6.—To mark goods so that the merchant may deduct a certain per cent. from the marked price and yet sell the goods at cost or at a certain per cent. above or below cost.*

(a) To sell at cost,

Ex. 1. Bought broadcloth at \$4.50 per yard; how shall I mark it so that I may deduct 10 per cent. from the marked price and yet sell at cost?

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* To determine the *per cent.* by which to increase the first cost, by an Algebraic process, we have this

RULE.—*Multiply the per cent. to be deducted from the marked price by* 100, *and divide the product by* 100 *diminished by the same per cent.; the quotient will be the per cent. by which to increase the first cost of the goods.*

Ex. Bought a case of boots at \$4 per pair; at what per cent. of increase shall I mark them to enable me to deduct 20 per cent. and yet sell them at cost?

$$\frac{100 \times 20}{100 - 20} = \frac{2000}{80} = 25, \text{ Ans.}$$

PROOF.—25 per cent. on \$4 is \$1, which added to \$4, gives \$5 for the marked price; again 20 per cent. on \$5 is \$1, which subtracted from \$5, the marked price, gives \$4, the cost.

Since the marked price is to be diminished by 10 per cent. of itself, the selling price must be the remaining 90 per cent. of the marked price; now if \$4.50 is 90 per cent., $\frac{1}{90}$ of $\$4.50 = 5$ cents is 1 per cent. and $\therefore$ 100 per cent. will be 100 times 5 cents $=$ \$5, Ans.

REMARK. — When the per centage can be reduced to a convenient vulgar fraction, as in Ex. 1, it may be better to employ the fraction rather than the per cent.; thus 90 per cent. or $\frac{90}{100} = \frac{9}{10}$, and if \$4.50 is $\frac{9}{10}$ of the marked price, $\frac{1}{9}$ of $\$4.50 = 50$ cents is $\frac{1}{10}$ and $\therefore$ $\frac{10}{10}$, i. e., the whole, will be 10 times 50 cents $=$ \$5, Ans. as before.

This example may also be solved by proportion; thus,

90 per cent. : 100 per cent. :: \$4.50 : \$5, Ans. as above. Hence,

RULE.— *As* 100 *diminished by the per cent. to be deducted is to* 100, *so is the cost to the marked price.*

2. Bought a shawl for \$18; how shall I mark it so that I may make a discount of 10 per cent. and yet sell it for \$18?

(b) To sell at a certain per cent. above or below cost,

Ex. 1. How shall I mark a watch which cost \$40 so that I may deduct 15 per cent. on the marked price, and yet make 25 per cent. on the cost?

Since I am to gain 25 per cent. or $\frac{1}{4}$ of the cost, I must sell it for $\$40 + \frac{1}{4}$ of $\$40 = \$50$.

Again, as I am to deduct 15 per cent., the \$50 is only $\frac{85}{100}$ of the marked price; hence the proportion: —

85 per cent. : 100 per cent. : : \$50 : $\$58.82\frac{6}{17}$, Ans.

In the above we first find the selling price, and then, treating that as the first cost, proceed as in Art. 271, a.

The example may also be solved by proportion; thus,

85 per cent. : 125 per cent. : : \$40 : $\$58.82\frac{6}{17}$, Ans. as before.

All similar examples may be solved in like manner. Hence,

RULE.—*Say as* 100 *diminished by the per cent. to be deducted is to* 100 *increased by the per cent. to be gained or diminished by the per cent. to be lost, so is the cost to the marked price*

2. Bought a piece of broadcloth at \$5, but, it being damaged, I am willing to lose 20 per cent.; how shall I mark it so that I may deduct 10 per cent. from the marked price?

$$90 : 80 :: \$5 : \$4.44\tfrac{4}{9}, \text{ Ans.}$$

3. Paid \$4 a pair for a case of boots; how shall I mark the same so that I may fall 10 per cent. from the marked price and yet make 25 per cent. on the cost?

4. Paid \$8 each for a case of bonnets; how shall I mark the same so that I may fall 12 per cent. from the marked price and yet make 10 per cent. on the cost?

5. Bought a pair of oxen for \$150; what shall I ask for them so that I may fall 5 per cent. and yet make 5 per cent. on the cost?

6. Bought a horse for \$175; what shall I ask for it so that I may fall 20 per cent. and yet lose but 5 per cent. on the cost?

272. MISCELLANEOUS EXAMPLES IN PROFIT AND LOSS.

1. Bought 10 tons of hay for \$155 and sold $\frac{1}{2}$ of it at \$15 per ton and the remainder at \$17; did I make or lose? how much?

2. What do I gain per cent. if I buy hats at \$3 and sell them at \$4.50 each?

3. Bought shoes at \$1.50 per pair and sold them at \$1.25; what per cent. on the cost was lost?

4. Sold potatoes at 75c. per bushel and lost 10 per cent. on the cost; for what should they be sold to gain 25 per cent.?

5. Paid \$3 per yard for a case of wrought laces; how shall I mark the same to enable me to make a discount of 25 per cent. from the marked price and yet gain 50 per cent. on the cost?

6. Sold cloth at \$4 per yard and lost 10 per cent.; should I have gained or lost, and how much per cent., if I had received \$4.25?

7. Sold a watch for \$40 and lost 12 per cent. on the cost; what was the cost?

8. A merchant bought broadcloth that was $1\frac{7}{8}$ yards wide for \$3.50 per yard, but the cloth, getting wet, shrunk 5 per cent. in width and 10 per cent. in length; at what price per square yard must it be sold to gain 5 per cent. on the cost?

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§ 30. PARTNERSHIP.

273. PARTNERSHIP is the association of two or more persons in business.

The company thus formed is called a *firm* or *house.*

The money or other property invested is called the *capital* or *stock* of the company.

The members of the firm are called stockholders.

The profits distributed from time to time among the stockholders are called *dividends.*

274. Ex. 1. A and B trade in company; A furnishes \$500 and B \$700; they gain \$480. How shall the partners share the gain?

Since A furnishes $\frac{5}{12}$ of the stock, he is entitled to $\frac{5}{12}$ of the gain, i. e. $\frac{5}{12}$ of \$480 = \$200; and, for a like reason, B is entitled to $\frac{7}{12}$ of the gain = \$280.

(a) Or we may make a proportion; thus,

Whole stock	:	A's stock	: :	Whole gain	:	A's gain.
\$1200	:	\$500	: :	\$480	:	\$200;
and Whole stock	:	B's stock	: :	Whole gain	:	B's gain.
\$1200	:	\$700	: :	\$480	:	\$280. Hence,

To find the respective shares of gain or loss,

RULE 1.—*Multiply the total gain or loss by each partner's fractional part of the stock, and the products will be the respective shares of gain or loss;* or,

RULE 2.—*Say, as the whole stock is to each partner's share of the stock, so is the total gain or loss to his share of the gain or loss.*

275. PROOF.—*The sum of the shares of gain or loss must equal the total gain or loss.*

Ex. 2. A, B and C form a partnership; A furnishes $3000, B $5000 and C $7000; they gain $3000. How shall the gain be divided? Ans. A's, $600; B's, $1000; C's, $1400.

3. Had the firm in Ex. 2 lost $600, what part of the loss must each partner sustain? how many dollars?

1st Ans. A, $\frac{1}{5}$; B, $\frac{1}{3}$; C, $\frac{7}{15}$.
2d Ans. A, $120; B, $200; C, $280.

4. A, B and C trade with a joint capital; A furnishes $\frac{1}{3}$ of the stock; B, $\frac{2}{7}$; and C the remainder; they gain $4284.21; what is each partner's share of the gain?

NOTE.—These rules are equally applicable to distributing the property of a bankrupt and many other similar problems.

5. A bankrupt whose property is worth $3000 owes A $2000, B $1500 and C $1000; to what fractional part of the property is each creditor entitled? to how many dollars?

6. A, B and C hire a pasture, for which they pay $150; A pastures 10 oxen; B, 8, and C, 12; what part of the rent shall each pay? how many dollars?

7. A and B hire a pasture for $15; A's horse was in the pasture $7\frac{3}{7}$ weeks and B's $22\frac{4}{7}$ weeks; what rent shall each pay?

8. A, B, C and D freight a ship to California; A furnishes $12000 worth of the cargo; B, $9000; C, $14000, and D, $15000; they gain $25000. What is each one's share of the gain?

9. Divide $800 between A, B and C so that A shall receive $2 as often as B receives $5 and C $9.

10. A, B and C hire a pasture for \$523.80 and stock it with horses, oxen, cows and sheep; 6 sheep are reckoned as 1 cow, 5 cows as 3 oxen and 5 oxen as 4 horses. A put in 3 horses, 6 oxen, 8 cows and 10 sheep; B, 2 horses, 3 oxen, 2 cows and 40 sheep; C, 4 horses, 5 oxen, 4 cows and 50 sheep. What shall each man pay?

11. A father proposed to divide \$1000 between his two sons in the ratio of $\frac{1}{3}$ to $\frac{1}{2}$; what was the share of each?

Ans. 1st son, \$400; 2d son, \$600.

12. If \$500 be divided between A, B and C in the proportion of $\frac{2}{3}$, $1\frac{1}{4}$ and $2\frac{1}{2}$ respectively, what will be the share of each?

13. A gentleman, dying, left two sons and a daughter, to whom he bequeathed \$2000, \$1500 and \$1000 respectively; but his whole estate sold for only \$2700 above debts and costs of settlement. What did each child receive from the estate?

14. A and B form a partnership with a joint capital of \$1000, of which A furnishes $\frac{2}{3}$ in cash, and B, for his share, furnishes 100 yards of broadcloth. They gain $\$333.33\frac{1}{3}$. How shall the profits be divided? What is the price of B's cloth per yard?

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§ 31. COMPOUND PARTNERSHIP.

276. Compound Partnership is a partnership in which the shares of stock are in for *unequal periods of time.*

Ex. 1. A and B trade in company; A puts in \$300 for 8 months, and B \$400 for 7 months. They gain \$156. What part of the gain belongs to each? How many dollars?

A's \$300 for 8m. = \$2400 for 1m.
B's \$400 for 7m. = \$2800 for 1m.
\$5200 for 1m.;

it is, ∴ as though the joint stock were \$5200 for 1 month, of which A put in \$2400, and B \$2800; hence A is entitled to

$\frac{2400}{5200} = \frac{6}{13}$ of the gain, and B to $\frac{2800}{5200} = \frac{7}{13}$; i. e. A is entitled to $\frac{6}{13}$ of \$156 = \$72, and B to $\frac{7}{13}$ of \$156 = \$84.

(a) The examples in this article may not only be solved as the above, but also by proportion by the following

RULE. — *Multiply each man's stock by the time it is continued in trade; then say, as the sum of all the products is to each man's product, so is the total gain or loss to each man's gain or loss, respectively.*

Thus, Ex. 1, above,

\$5200 : \$2400 :: \$156 : \$72, A's gain, and
\$5200 : \$2800 :: \$156 : \$84, B's gain.

2. A and B traded in company; A furnished \$1200 for 8 months, and B \$1700 for 11 months. They lost \$500. What was the loss of each?

3. Jan. 1, 1853, A, B and C form a partnership for 1 year, and each furnishes \$3000; Mar. 1, A furnishes \$1000 more; June 1, B withdraws \$500, and C adds \$500; Sept. 1, A withdraws \$2000 and C \$500, and B adds \$1500. Having gained \$4000, at the close of the year the partnership is dissolved. What is each partner's share of the gain?

4. A, B and C traded in company. A, at first, put in \$1000, B \$1200 and C \$1800; in 3 months, A put in \$500 more and B \$300, and C took out \$400; in 7 months from the commencement of business, A withdrew all his stock but \$700, B put in as much as he at first put in and C withdrew $\frac{1}{3}$ as much as A at any time had in the firm. At the end of a year, they found they had gained 10 per cent. on the largest total stock at any one time in trade. What is the total gain? What fractional part shall each have? How many dollars?

Ans. Total gain, \$440. { A's part, $\frac{125}{511}$ = \$107.63$\frac{107}{511}$.
B's part, = \$
C's part, = \$
Proof, = \$

5. A, B and C hire a pasture for \$300. A puts in 10 oxen for 20 weeks, 15 cows for 14 weeks, and 99 sheep for 26 weeks; B puts in 7 oxen for 24 weeks, 12 cows for 20 weeks, and 66 sheep for 25 weeks; C puts in 25 oxen for 8 weeks, 12 cows for 12 weeks, and 33 sheep for 15 weeks. Now, if 11 sheep are reckoned as 1 cow, and 3 cows as 2 oxen, what is the cost per week for a sheep? a cow? an ox? How many dollars does each man pay for sheep? cows? oxen? What part of the rent does each man pay? How many dollars?

Ans. Cost per week for a sheep, $1\frac{5}{11}$c; a cow, 16c.; an ox, 24c. A pays for sheep, \$37.44; for cows, \$33.60; for oxen, \$48. B pays for sheep, \$24; for cows, \$38.40; for oxen, \$40.32. C pays for sheep, \$7.20; for cows, \$23.04; for oxen, \$48. A pays $\frac{248}{625}$ = \$119.04; B, $\frac{214}{625}$ = \$102.72; C, $\frac{163}{625}$ = \$78.24.

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§ 32. TAXES.

277. A TAX is a duty levied upon the person or the property of individuals by the authorities of a town, county, state, or other section of a country, or by the national government, to defray the expenses of government, to construct public works of common utility, etc.

278. The tax levied upon the *person* * is called the *capitation* † or *poll* ‡ *tax*, and is so much to each individual liable to pay a poll tax, without any reference to his property.

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* In Massachusetts, males 20 years of age and upward are subject to pay a poll tax. From this rule paupers are exempt, and aged and infirm men sometimes have the poll tax remitted. The payment of a poll tax is a requisite to the privilege of exercising the elective franchise.

† *Capitation*, from the Latin *caput, the head.*

‡ *Poll*, from the Dutch *bol, the head.*

279. The method of assessing taxes is not the same in all its details in the different States, but the essential principles are the same.

280. In Massachusetts, the assessors are required to assess upon the polls about one-sixth part of the tax to be raised, provided the poll tax of one individual for town and county purposes, except highway taxes, shall not exceed \$2.00 for one year. The remainder of the sum to be raised is apportioned upon the taxable property of the town, county or state. Hence,

To Assess Taxes,

RULE.—*Ascertain the number of polls liable to taxation and take an inventory of the taxable property. Multiply the sum assessed upon one poll by the number of taxable polls and subtract the product from the sum to be raised. Divide the remainder by the taxable property and the quotient will be the tax upon* \$1. *Multiply the tax upon* \$1 *by the taxable property of an individual; to the product add his poll tax and the sum will be his tax.*

Ex. 1. The town of A is to be taxed \$5999. The real estate of the town is valued at \$500000 and the personal at \$300000. There are 666 taxable polls, each of which is assessed \$1.50. What is the tax of B, whose real estate is valued at \$4000 and his personal property at \$8000, and who pays 1 poll tax?

$\$1.50 \times 666 = \$999$, sum assessed on the polls.
$\$5999 - \$999 = \$5000$, sum to be assessed on the property.
$\$500000 + \$300000 = \$800000$, amount of taxable property
$\$5000 \div 800000 = 6\frac{1}{4}$ mills, tax on \$1.
$\$4000 + \$8000 = \$12000$, B's taxable property.
$\$12000 \times .006\frac{1}{4} = \$75$, tax on B's property.
$\$75 + \$1.50 = \$76.50$, B's entire tax, Ans.

2. The town of F, wishing to raise a tax of \$3599.20, has real estate valued at \$350000 and personal property worth \$250000. There being 428 polls, each of which is assessed at \$1.40, what

is the tax of E, whose real estate is valued at $4000 and his personal property at $6500, and who pays for 3 polls?

3. A's property is valued at $5000, and he pays for 2 polls.
B's " " " 2400, " " 3 "
C's " " " 3600, " " 1 "
D's " " " 1690, " " 2 "

What are their respective taxes, the conditions being the same as in Ex. 2?

NOTE.—To save labor, (by using smaller numbers,) assessors frequently take 6 per cent. of the inventory of the town and of the several inhabitants, instead of the entire valuation; but the labor may be lessened still more by taking 1 or 10 per cent.

281. It will be more convenient, in calculating a tax list, first to form a table showing the tax on $1, $2, $3, etc., in the percentage column, and then to calculate the individual taxes from the table.

Ex. 4. The town of W, whose valuation is $666800, has 10 taxable inhabitants, A, B, C, etc., and wishes to raise a tax of $3358.

The taxes of the several inhabitants are for the number of polls and the property, as in the annexed

TABLE.

Names.	No. Polls.	Real Estate.	Personal Estate.	Total.	10 per cent.
A	3	$75596	$24404	$100000	$10000
B	2	13846	36154	50000	5000
C	2	75000		75000	7500
D	1	18544	41456	60000	6000
E	2	24692	122358	147050	14705
F		27650	31500	59150	5915
G			50000	50000	5000
H	2	15000	20300	35300	3530
I	1	20000	39950	59950	5995
J	3	10125	20225	30350	3035
Total	16	280453	386347	666800	66680

The tax upon each poll being $1.50, what are the respective taxes of A, B, C, etc.?

In solving this question, first find the sum of all the poll taxes, ($\$1.50 \times 16 = \$24$,) and, having deducted this from the total tax, ($\$3358 - \$24 = \$3334$), divide the remainder by the taxable property in town, ($\$3334 \div 66680 = \$.05$), to find the tax on $1; then form a table showing the tax on $1, $2, $3, etc.; thus,

TABLE.

$		$	$		$	$		$
1	gives	0.05	40	gives	2.00	700	gives	35.00
2	"	0.10	50	"	2.50	800	"	40.00
3	"	0.15	60	"	3.00	900	"	45.00
4	"	0.20	70	"	3.50	1000	"	50.00
5	"	0.25	80	"	4.00	2000	"	100.00
6	"	0.30	90	"	4.50	3000	"	150.00
7	"	0.35	100	"	5.00	4000	"	200.00
8	"	0.40	200	"	10.00	5000	"	250.00
9	"	0.45	300	"	15.00	6000	"	300.00
10	"	0.50	400	"	20.00	7000	"	350.00
20	"	1.00	500	"	25.00	8000	"	400.00
30	"	1.50	600	"	30.00	9000	"	450.00

Now to find I's tax from this table:—

I's tax on	$5000	=	$250.
"	900	=	45.
"	90	=	4.50
"	5	=	.25
"	$5995	=	$299.75
I's poll tax		=	1.50
	I's total tax	=	$301.25.

In a similar manner each one's tax may be found.

Ex. 5. The town of C, whose taxable property amounts to $1058000, wishes to raise a tax of $3184. There are 385 polls, each taxed $1.40. The property of

A is valued at $\$33333.33\frac{1}{3}$, and he pays for 2 polls;

B's	is	valued at	$6666.66⅔	and	he pays for	3	polls;
C's	"	"	16666.66⅔	"	"	1	"
D's	"	"	50000.	"	"	2	"
E's	"	"	25000.	"	"	4	"
F's	"	"	75000.	"	"	2	"
G's	"	"	15533.33⅓.	"	"	1	"

What percentage of tax is laid upon the property and what is the tax of A and of each on the list? Ans. ¼ of 1 per cent.

A's tax $86.13⅓
B's " 20.86⅔
C's " etc.

282. In Connecticut, personal property is taxed just twice as high as real estate; thus, if A pays $30 on a farm worth $4000, then B would pay $60 on $4000 at interest.

283. In Vermont, each taxable poll is reckoned as so much property, say $200, and no separate poll tax is calculated. This shortens the operation of making out a tax list, and is, virtually, the same as in Massachusetts.

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§ 33. ALLIGATION.

284. Alligation* treats of mixing simple substances of different qualities, producing a compound of some intermediate quality. It is of two kinds, *Medial* and *Alternate*.

285. Alligation Medial is the process by which we find the price of the mixture, when the quantities and prices of the simples are given.

Ex. 1. A merchant mixes 5 gallons of oil worth 4s. per gal.

**Alligation*, from the Latin *alligo, to bind or unite one thing to another;* a name suggested by the *mode* of operation in the *linking process.*

with 4 gal at 5s., 2 gal. at 11s. and 3 gal. at 12s. What is the value of a gallon of the mixture?

5	gal. at 4s.	per gal.	are worth	20s.
4	" 5s.	"	"	20s.
2	" 11s.	"	"	22s.
3	" 12s.	"	"	36s.
∴ 14	gal.		are worth	98s.

and 1 gal. is worth $\frac{1}{14}$ of 98s. = 7s., Ans.

All examples of this nature are solved on this plan. Hence,

RULE.—*Divide the total value of the articles mixed by the sum of the simples, and the quotient is the price of* ONE.

Ex. 2. A miller mixes 75 bushels of corn worth $1.05 per bush. with 25bush. barley at $1.20, 5bush. rye at $1.50 and 20bush. wheat at $2; what is the value of a bushel of the mixture? Ans. $1.25.

3. A grocer mixes 5lb. sugar worth 4c. per lb. with 4lb. at 6c., 2lb. at 9c., 2lb. at 11c. and 4lb. at 13c.; what is a pound of the mixture worth?

4. If I mix 23lb. of spice worth 12c. per lb. with 7lb. at 20c., 5lb. at 25c., 8lb. at 38c., 13lb. at 40c. and 21lb. at 56c.; what is a pound of the mixture worth?

286. ALLIGATION ALTERNATE is the process of mixing quantities of different prices so as to obtain a mixture of a required intermediate price.

There are three problems.

287. PROB. 1.—The prices of several kinds of goods being given to ascertain how much of each kind may be taken to form a compound of a proposed medium price.

Ex. 1. A farmer wishes to mix oats worth 30c. per bush. with barley worth 45c., so as to make a mixture worth 42c.; how many bushels of each may he take?

There are several ways of solving this question.

(a) It is evident that he must mix them in such proportions as to gain just as much on his oats as he loses on the barley Now, he gains 12c. on 1bush. of oats and loses but 3c. on 1bush of barley ; ∴, for each bushel of oats he must take 4 bushels of barley.

SECOND METHOD.

$$\text{(b)}\quad 42\begin{cases}30\;\rceil & 3\\ 45\;\rfloor & \mathbf{12}\end{cases}\qquad \begin{matrix}-\,12\text{c.}\times\;\;3=-36\text{c., deficiency.}\\ +\;\;3\text{c.}\times12=+36\text{c., surplus.}\end{matrix}$$

Having written the prices of the oats and barley in a vertical column and the price of the mixture at the left, as above, we write the difference between the mean price (i. e. the price of the mixture) and the price of the oats against the price of the barley, and the difference between the mean price and that of the barley against the price of the oats, and the differences standing against the prices of the oats and barley, respectively, will represent the *proportional quantities* of oats and barley to be taken ; for it will be seen that the product of the deficiency in the value of a bushel of oats multiplied by the number of bushels of oats ($-$ 12c. $\times$ 3 $=$ $-$36c.) is *necessarily* equal to the product of the surplus in the value of a bushel of barley multiplied by the number of bushels of barley ($+$ 3c. $\times$ 12 $=$ $+$36c.), *since the two products are composed of the* SAME FACTORS ; and, one representing a deficiency and the other a surplus, *they will balance each other.*

In the same manner, any number of *pairs* of simples may be made to balance, as in Ex. 2, 2d method, the price of one simple in each pair being *less* and that of the other *greater* than the mean price.

In performing the operation, the terms are connected together by a line merely for convenience of reference in comparing them.

288. Any multiples or sub-multiples of these numbers will be of the same value per bushel.

Ex. 2. I have oats at 40c. per bush., barley at 47c., corn at 95c. and rye at 98c.; how may I mix them so as to make a mixture worth 75c. per bushel?

1 bush.	oats,	—35c.	
1 "	barley,	—28c.	
	Deficiency,		—63c.
1 "	corn,	+20c.	
$1\frac{20}{23}$ "	rye,	+43c.	
	Surplus,		+63c.

1 bushel of oats gives a deficiency of 35c. and 1 of barley of 28c.; 1 bushel of corn gives a surplus of 20c., which partially cancels the deficiencies, but still leaves a deficiency of 43c.; now, 1bush. of rye gives a surplus of 23c.; ∴ $1\frac{20}{23}$bush. of rye will balance the deficiency of 43c. and the mixture will be worth 75c. per bushel.

289. This mode of analyzing is liable to fail if we take the prices in regular order, beginning with the lowest; thus, had the prices of the several kinds of grain been as in Ex. 2, and the price of the mixture 50c., then it will be seen that, having taken 1bush. each of oats, barley and corn, we cannot restore the balance by putting in rye, for the mixture is already of too high a price (the surplus, 45c., being greater than the deficiency, 13c.), and adding rye will increase the price; however, in this example, we may begin with the highest price, and so remove the difficulty, and in every example we may begin with the lowest *or* with the highest price, and obtain a correct result.

1bush. oats,	—10c.
1bush. barley,	— 3c.
Deficiency,	—13c.
1bush. corn,	+45c.

SECOND METHOD.

		bush.			
75	40	23	—35c. × 23 = —805c.		
	47	20	—28c. × 20 = —560c.	—1365c.	
	95	28	+20c. × 28 = +560c.		
	98	35	+23c. × 35 = +805c.	+1365c.	

Each *pair* of these products, viz. the 1st and 4th, and the 2d and 3d, will necessarily balance; *for they are composed of the* SAME FACTORS, and one is + and the other —.

290. There evidently may be as many independent answers, all correct, as there are different ways of *pairing* the simples; and, by taking multiples and sub-multiples of these, the results may be varied indefinitely, so that there may be an infinite number of answers to one question.

Among other methods, the 2d example may have the following solutions, and each solution may be *proved* correct by *Alligation Medial.*

$$75\left\{\begin{array}{l}40\\47\\95\\98\end{array}\right. \quad \begin{array}{l}20 \text{ at } 40\text{c.}\\23 \text{ “ } 47\text{c.}\\35 \text{ “ } 95\text{c.}\\28 \text{ “ } 98\text{c.}\end{array} \qquad\qquad 75\left\{\begin{array}{l}40\\47\\95\\98\end{array}\right. \quad \begin{array}{r}20+23=43 \text{ at } 40\text{c.}\\23 \text{ “ } 47\text{c.}\\35 \text{ “ } 95\text{c.}\\35+28=63 \text{ “ } 98\text{c.}\end{array}$$

$$75\left\{\begin{array}{l}40\\47\\95\\98\end{array}\right. \quad \begin{array}{r}20 \text{ at } 40\text{c.}\\20+23=43 \text{ “ } 47\text{c.}\\35+28=63 \text{ “ } 95\text{c.}\\28 \text{ “ } 98\text{c.}\end{array}$$

From these illustrations,

RULE. — *Write the prices of the several simples in a vertical column; on the left, separated by a line, write the proposed medium price; connect, by a line, each price that is less than the medium with one or more that is greater, and each that is greater with one or more that is less; write the difference between the medium price and the price of each simple against the number or numbers with which the simple is connected; these differences, or their sum if two or more stand against one price, will be the proportional parts of the several simples which may be taken to form the mixture.*

291. To the mathematician there is something satisfying in the analytic process, and something pleasing in the balancing of deficiencies and excesses by the linking process; but, for the merchant's convenience, there is yet another mode, which may, with propriety, be denominated the *Yankee Method*, viz. that of *guessing* at the quantities of the several simples, and then, by calculation, adjusting the guess.

To illustrate, let us resume Ex. 2.

$$
\begin{array}{ll|rcrcrl}
 & \text{cts.} & \text{bush.} & & \text{cts.} & & \text{cts.} & \\
\text{cts.} & 40 & 5 & \times & -35 & = & -175 & \\
75 & 47 & 8 & \times & -28 & = & -224 & \text{cts.} \\
 & & & & & & & -399, \text{ deficiency.} \\
 & 95 & 6 & \times & +20 & = & +120 & \\
 & 98 & 4 & \times & +23 & = & +\ 92 & \\
 & & & & & & & +212, \text{ surplus.} \\
 & & & & & & & -187, \text{ deficiency.}
\end{array}
$$

$$
\begin{array}{lrcrcl}
 & \text{bush.} & & \text{cts.} & & \\
\text{Add rye,} & 9 & \times & +23 & = & +207, \text{ surplus.} \\
 & & & & & +20, \text{ surplus.} \\
\text{Subtract corn,} & -1 & \times & +20 & = & -20, \text{ deficiency.} \\
 & & & & & \quad 0
\end{array}
$$

Having assumed 5 bushels of oats, 8 of barley, 6 of corn and 4 of rye, we find the mixture is not worth so much as it should be by $1.87. Now, this may be remedied by putting in more of the higher priced grains or less of the cheaper. If we add 9bush. more of the rye, this will balance the deficiency and create a surplus of 20cts. and this may be corrected by taking out 1bush. of corn. There will now be in the mixture 5bush. of oats, 8 of barley, 5 of corn and 13 of rye.

REMARK.—The deficiencies are marked by the sign — and the excesses by + to aid the mind in making corrections.

NOTE.—This mode of correcting may be indefinitely varied, hence the merchant may take the simples in a ratio more nearly as he desires than by either of the other modes.

292. Let the pupil solve the following examples by each of the three modes and prove them :—

3. A merchant has 4 kinds of oil worth 3s., 5s., 10s. and 13s. per gallon. What quantities of each may he take to make a mixture worth 7s. per gallon?

4. A grocer wishes to mix teas worth 20c., 28c., 33c., 47c. and 60c. so that the compound may be worth 38c. per pound. How many pounds of each may he take?

5. A grocer wishes to mix water of no exchangeable value with wines worth \$1.50, \$1.80 and \$2 per gallon. How many gallons of each may he take to make a mixture worth \$1.60 per gallon?

6. A grocer has spices worth 4, 7, 11, 15 and 20 cents per lb. How many pounds of each may he take to make a mixture worth 12 cents per lb.?

293. Prob. 2.—The price of each of the simples, the price of the compound and the quantity of one kind being given, to find how much of each of the other simples may be taken,

Rule.—*Find the proportional parts as in the preceding case; then say, as the proportional part of that simple whose quantity is given is to the given quantity, so is each of the other proportional parts to the required quantity of each of the other simples, severally.*

Or, having found the proportional parts, the question may be analyzed.

Ex. 1. How many pounds of sugar at 4, 6, 9 and 10c. per lb. may be mixed with 12lb. at 13c. so as to make a compound worth 8c. per lb.?

	cts.		lbs.
cts. 8	4		5
	6	2 + 1 =	3
	9		2
	10		2
	13		4

If we connect the prices as in the margin, we obtain 5, 3, 2, 2 and 4lb. for the proportional parts. Now if the 4lb. at 13c. be increased in a 3 fold ratio, it will become 12lb., *the given quantity*, and if each of the other proportional parts be increased in the same ratio, evidently the price per lb. of the mixture will remain unaltered; hence,

4lb. at 13c. : 12lb. at 13c. : : 5lb. at 4c. : 15lb. at 4c.
4lb. at 13c. : 12lb. at 13c. : : 3lb. at 6c. : 9lb. at 6c.
etc. etc.

Ans. 15, 9, 6 and 6lb. at 4, 6, 9 and 10c.

Ex. 2. How many gallons of wine at 6, 9 and 15s. per gal. may be mixed with 40gal. of water to make a compound worth 10s. per. gal.?

3. How many gallons of molasses at 20, 25 and 33c. per gal. may be mixed with 63gal. at 50c. to make a compound worth 30c. per gal.?

4. How many pounds of wool at 24, 30 and 40c. per lb. may be mixed with 100lb. at 50c. to form a mixture worth 37c. per lb.?

5. How many ounces of gold that is 16, 18 and 22 carats fine may be mixed with 15oz. that is 24 carats fine to form a mixture 21 carats fine?

294. Prob. 3.—The prices of the several simples, the price of the compound and the entire quantity in the compound being given, to find how much of each simple may be taken,

Rule. — *Find the proportional parts as in Prob.* 1; *then say, as the sum of the proportional parts is to the whole compound, so is each of the proportional parts to the required quantity of each.*

Or, analyze.

Ex. 1. I have 4 kinds of coffee, worth 8, 11, 14 and 20c. per pound; how many pounds of each may I take to form a compound of 60lb. at 13c. per lb.?

Ans. 28, 4, 8 and 20lb. at 8, 11, 14 and 20c.

	cts.	lbs.
cts. 13	8	7
	11	1
	14	2
	20	5
		15

15 : 60lb. : : 7lb. : 28lb. at 8c.
15 : 60lb. : : 1lb. : 4lb. at 11c.
etc. etc.

We find that the sum of the proportional parts, if linked as above, is 15lb., and if this be quadrupled, 60lb., the required compound, will be obtained; but the whole compound will be quadrupled by increasing each of the proportional parts in a four fold ratio.

2. A merchant has spices, worth 25, 31, 40, 42, 45, 50 and 70c. per lb.; how many pounds of each may he take to form a compound of 300lb. at $37\frac{1}{2}$c. per lb.?

3. A merchant mixes water with wines, worth 75, 90, 100 and 124c. per gal. so as to make a mixture of 3000gal. worth $1.08 per gal.; how many gal. of each may he take?

4. A drover has sheep, worth 9, 10, 15, 18 and 24s. each how many of each may he take to form a flock of 160 sheep worth 16s. each?

5. How many ounces of gold that is 18, 20, 23 and 24 carats fine, may be taken to form a mass of 30 ounces, that shall be 21 carats fine?

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§ 34. SINGLE POSITION.

295. Single Position is a method of solving an analytical question by assuming a number and working with it as though it were the true answer to the question.

296. Rule. — *Assume any number and proceed with it according to the conditions of the question; then say, as the result obtained is to the result given in the question, so is the assumed number to the required number.*

Ex. 1. What number is that, which, being increased by $\frac{1}{2}$ and $\frac{1}{3}$ of itself, the sum will be 44?

$$\begin{array}{ll} \text{Assume} & 6 \\ \text{Add } \frac{1}{2} \text{ of } 6 = & 3 \\ \text{Also } \frac{1}{3} \text{ of } 6 = & \underline{2} \\ \text{Result,} & 11 : 44 :: 6 : 24, \text{ Ans.} \end{array}$$

Had we assumed 24, the true number, our result would have been 44, whereas it is only 11; however, we have performed the same operations on 6 to obtain 11, that we should have performed on 24 to obtain 44; ∴ the proportion, 11 : 44 :: 6 : 4th term, must give the number sought.

All examples in Single Position may be very easily analyzed, thus in Ex. 1, the number sought is $\frac{6}{6}$ of itself; $\frac{1}{2}$ the number is $\frac{3}{6}$ and $\frac{1}{3}$ is $\frac{2}{6}$; now $\frac{6}{6}+\frac{3}{6}+\frac{2}{6}=\frac{11}{6}$, i. e. 44 is $\frac{11}{6}$ of the number. If 44 is $\frac{11}{6}$ of the number, then $\frac{1}{11}$ of 44 is $\frac{1}{6}$; $\frac{1}{11}$ of 44 is 4, and if 4 is $\frac{1}{6}$, then $\frac{6}{6}$, or the whole number, is 6 times $4=24$, Ans. as before.

Let the learner solve the following examples both by Analysis and Position:—

2. Divide 540 into 3 such parts that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second and $\frac{1}{4}$ of the third shall be equal to each other.

BY POSITION.

Let 20 = 1st part
then 30 = 2d "
and 40 = 3d "

90 : 540 : : 20 : 120, 1st part
90 : 540 : : 30 : 180, 2d "
90 : 540 : : 40 : 240, 3d "

Proof, 540.

Though it is right to assume any number whatever, yet it is usually more convenient to assume such numbers as will avoid fractions.

BY ANALYSIS.

The smaller number is $\frac{2}{2}$ of itself, and, by the nature of the question, the 2d number is $\frac{3}{2}$ of the 1st and the 3d is $\frac{4}{2}$ of the 1st; $\therefore$ 540 is $\frac{2}{2}+\frac{3}{2}+\frac{4}{2}=\frac{9}{2}$ of the 1st; if 540 is $\frac{9}{2}$, then $\frac{1}{9}$ of 540 is $\frac{1}{2}$; $\frac{1}{9}$ of 540 is 60, and if 60 is $\frac{1}{2}$ of the 1st number, then $\frac{2}{2}$, or the whole, is twice $60=120$, 1st number.

3. A teacher, being asked how many scholars he had, replied, if I had as many more, $\frac{1}{2}$ and $\frac{1}{3}$ as many more and $36\frac{1}{2}$ scholars, I should have 300; how many had he? Ans. 93.

4. A and B have the same income; A saves $\frac{1}{3}$ of his, but B, by spending twice as much as A, at the end of 4 years, finds himself \$480 in debt; what is the annual income of each?

Ans. \$360.

5. A man, being asked his age, replied, if $\frac{2}{5}$ of the years I have lived be multiplied by 9 and $\frac{3}{5}$ of them be subtracted from the product, the remainder will be 150. How old was he?

6. Seven eighths of a certain number exceed $\frac{1}{5}$ of the same by 81; what is the number? Ans. 120.

7. A man lent a sum of money at 6 per cent., compound interest, and at the end of 3 years received the amount, \$11910.16; what was the interest?

8. A gentleman bought a chaise, horse and harness for \$470; the horse cost $\frac{5}{4}$ as much as the harness and the chaise $\frac{4}{3}$ as much as the horse; what was the price of each?

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§ 35. DOUBLE POSITION.

297. Double Position is a method of solving an analytical question by assuming *two* numbers and working with each as though it were the true answer to the question.

298. Rule.—*Assume any two numbers and proceed with each as the conditions of the question require; compare each result with the result given in the question and call each difference an error; multiply the* 1*st assumed number by the* 2*d error and the* 2*d assumed number by the* 1*st error; then if both assumed numbers are too great or both too small, divide the difference of the products by the difference of the errors; but, if one assumed number is too great and the other too small, divide the sum of the products by the sum of the errors; in either case, the quotient will be the number sought.*

Ex. 1. A gentleman, having a sum of money, spent \$100 more than $\frac{1}{5}$ of it and had remaining \$35 more than $\frac{1}{2}$ of it; how much had he at first?

Suppose, 1st, he had	\$1000, then,	
\$100 more than $\frac{1}{5}$ of it =	\$300	= the sum spent,
and	\$700	= sum remaining;
but \$35 more than $\frac{1}{2}$ of it =	\$535,	
	$\therefore +$ \$165	= 1st error.

Suppose, 2d, he had $1500, then,
$100 more than $\frac{1}{5}$ of it = $400 = the sum spent,
and $1100 = the sum left;
but $35 more than $\frac{1}{2}$ of it = $785,
$\therefore +$ $315 = 2d error.

$1000 × 315 = $315000, i. e. 1st assumed No. × 2d error.
$1500 × 165 = $247500, i. e. 2d assumed No. × 1st error.
$67500 ÷ 150 = $450, Ans.;

i. e. the difference of the products divided by the difference of the errors gives $450, the answer.

REMARK.—This rule is applicable to all examples that can be solved by Single Position, and also to very many problems usually solved by Algebra. It is founded on the supposition that the 1st error is to the 2d error as the difference between the true and 1st supposed number is to the difference between the true and 2d supposed number. When this proportion does not hold, the problem cannot be solved directly by the rule.*

NOTE —Let the pupil solve the following examples, both by Position and Analysis.

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* ALGEBRAIC DEMONSTRATION OF THE RULE.—Having assumed the numbers a and b, and performed on them the operations required by the conditions of the example, let the results be represented by A and B, whereas, if we had assumed the true number, x, we should have obtained N, the result given in the example.

Now let $N - A = r$, the 1st error, and
$N - B = s$, the 2d error;
Then by the proportion in Art. 298, Remark,
We have $r : s :: x - a : x - b$
Reducing to an equation, $rx - rb = sx - sa$
Transposing, $rx - sx = rb - sa$
Dividing by $r - s$, $x = \dfrac{rb - sa}{r - s}$

Had r AND s both been *negative*, the value of x would not have been changed. Had r OR s been negative, the proportion would have taken one of two following forms, $-r : s :: x - a : x - b$, or $r : -s :: x - a : x - b$, either of which, reduced, will give $x = \dfrac{rb + sa}{r + s}$; and these values of x agree with the enunciation of the rule.

2. A and B have the same income; A saves $\frac{1}{5}$ of his, but B, by spending £30 per annum more than A, at the end of 8 years finds himself £40 in debt. What is their annual income?

Ans. £200 each.

3. A wine dealer bought 2 casks of porter, one of which held 3 times as much as the other; from each of these he drew 4gal., when 4 times as many gal. remained in the one as in the other. Required the number of gal. in each. Ans. 12 and 36.

4. A and B have the same income; A saves $\frac{1}{3}$ of his, but B, by spending \$200 per annum more than A, finds himself in debt. At the end of 5 years, A lends to B enough to pay his debt and has \$250 left. What is the annual income of each?

5. What number is that which, being divided by 7, and the quotient diminished by 10, 3 times the remainder shall be 24?

Ans. 126.

6. There is a fish whose head weighs 14 pounds, his tail weighs as much as his head and $\frac{3}{10}$ as much as his body, and his body weighs as much as his head and tail. What is the weight of the fish? Ans. 80lb.

7. A man hired a laborer for 50 days, on condition that for every day he worked he should receive \$1.50 and for every day he was absent he should forfeit \$1.75. At the expiration of the time he received \$42.50. How many days was he absent?

Ans. 10.

8. A drover bought a number of horses, oxen and cows for \$2640. For every horse he paid \$50, for each ox $\frac{8}{5}$ as much as for a horse, and for each cow $\frac{1}{2}$ as much as for a horse. There were 3 times as many oxen as horses, and twice as many cows as oxen. How many were there of each?

9. A gentleman has 2 horses and a saddle. The saddle is worth $\frac{1}{2}$ as much as the 1st horse, and if it be put on the 1st horse, they together will be worth .3 as much as the 2d horse. If the saddle be put on the 2d horse they will be worth 3 times as much as the 1st horse plus \$50. What is the value of the 2d horse?

§ 36. INVOLUTION AND POWERS.

299. If a number is multiplied by itself, the product is called a power; thus,

$3 \times 3 = 9$, the 2d power, or square of 3.
$3 \times 3 \times 3 = 27$, the 3d power, or cube of 3.
$3 \times 3 \times 3 \times 3 = 81$, the 4th power, or biquadrate of 3.
$3 \times 3 \times 3 \times 3 \times 3 = 243$, the 5th power of 3.
etc. etc.

Again, $10 \times 10 = 100$, the square of 10.
$10 \times 10 \times 10 = 1000$, the cube of 10.
$10 \times 10 \times 10 \times 10 = 10000$, the 4th power of 10.
$10 \times 10 \times 10 \times 10 \times 10 = 100000$, the 5th power of 10.
etc. etc.

The number which is multiplied by itself is the 1*st power;* it is also the *root* of the other powers (94, b, Note 5).

300. The process of multiplying a number by itself, i. e. raising it to any required power, is called Involution.

301. Instead of actually performing the multiplication, we may *indicate* the power by placing an *exponent* or *index* at the right and a little above the root (94, b, Note 2); thus,

4×4 is written 4^2, and is read, the square of 4, or 2d power of 4.
$4 \times 4 \times 4$ is written 4^3 and is read, the cube of 4, or 3d power of 4.
etc. etc.

302. The exponent shows how many times the root is taken as a factor. Hence,

To involve a number to any required power,

Rule 1.—*Write the index of the power over the root;* or,

Rule 2.—*Multiply the number by itself, and (if a higher power than the second is required) multiply this product by the original number and so on until the root has been taken as a fac-*

tor as many times as there are units in the index of the required power.

NOTE.—These rules are applicable to every example that can occur in involution; but their application may be profitably modified in particular cases.

303. If we involve by Rule 2, the number of multiplications is always one less than the number of units in the index of the power; thus, *one* multiplication gives the *second* power, *two* multiplications give the *third* power, etc.

We may, however, more readily obtain a high power by omitting some of the intermediate powers; thus, $3^2 \times 3^2 = 3^4 = 81$, for $3^2 \times 3^2 = \overline{3 \times 3} \times \overline{3 \times 3} = 3 \times 3 \times 3 \times 3 = 3^4$; so also $3^2 \times 3^3 = 3^5 = 243$, for $3^2 \times 3^3 = \overline{3 \times 3} \times \overline{3 \times 3 \times 3} = 3 \times 3 \times 3 \times 3 \times 3 = 3^5$; and, *generally*, if any two or more powers of the same number be multiplied together, the product will be that power of the root indicated by the *sum* of the exponents of the factors.

304. The principle in 303 leads directly to the following:—

To involve a quantity that is already a power,

RULE. — *Multiply the index of the given number by the index of the power to which it is to be raised.*

Thus, the 3d power of 2^2 is 2^6, for $2^2 = 2 \times 2$, and the 3d power of 2×2 is $\overline{2 \times 2} \times \overline{2 \times 2} \times \overline{2 \times 2} = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6 = 64$.

Again, the 5th power of 4^3 is $4^{15} = 1073741824$; etc.

305. A vulgar fraction is involved by involving the numerator and denominator separately (127 Rule).

Before involving, the fraction should be reduced to its simplest form; thus,

the 3d power of $\frac{4}{6}$ is $(\frac{4}{6})^3 = (\frac{2}{3})^3 = \dfrac{2^3}{3^3} = \frac{8}{27}$; etc.

306. The number of decimal places in the power of a decimal fraction is equal to the number of decimal places in the

root multiplied by the index of the power (156, Rule); thus, the 3d power of .12 will contain 6 decimal places; for, $.12 \times .12 = .0144$ and $.0144 \times .12 = .001728$; i. e. $.12^3 = .001728$; etc.

307. All the powers of 1 are 1; for the continued product of any number of 1's is 1.

The powers of a number greater than unity, are greater than the root and the powers of a proper fraction are less than the root; thus, $3^2 = 9 > 3$; $(\frac{5}{2})^3 = \frac{125}{8} > \frac{5}{2}$; etc., but $(\frac{2}{3})^2 = \frac{4}{9} < \frac{2}{3}$; $(\frac{4}{5})^3 = \frac{64}{125} < \frac{4}{5}$; etc., etc.

308. To divide a power of any number by any other power of the same number, we have only to subtract the index of the divisor from that of the dividend; thus, $5^7 \div 5^3 = 5^4$, for $5^7 \div 5^3$

$$= \frac{5^7}{5^3} = \frac{5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5}{5 \times 5 \times 5} = 5 \times 5 \times 5 \times 5 = 5^4; \text{ etc.}$$

309. The product of two numbers cannot consist of more figures than there are in the two factors, counted together, nor of less than that number minus one; thus, take the largest numbers that can be expressed by 2 and by 3 figures, viz., 99 and 999. Now, since 999×99 is less than 999×100, and 999×100 ($= 99900$) contains only as many figures as are in 999 and 99, it is evident that the product of 999×99, or of any other two numbers consisting of but 2 and 3 figures, cannot have more than 5 figures; i. e. it cannot have more than the number of figures in the two factors.

Again, take the smallest numbers that can be expressed by 2 and by 3 figures, viz., 10 and 100; then $100 \times 10 = 1000$, and the product of any other two numbers expressed by 2 and 3 figures will be greater than 1000; but 1000 has only one figure less than 10 and 100 counted together, i. e. only one figure less than are in the two factors.

Like illustrations may be made with any two numbers; ∴ the truth of this article is evident.

310. An important application of this principle is, that the *square* of a number will always consist of *twice* as many figures

as the root or of *one less* than twice as many; the *cube* of a number will consist of *three times* as many as the root or of *one* or *two less* than three times as many; etc.

311. The square of units cannot consist of a higher order of figures than *tens*, for the square of 9, the largest unit's figure, is but 81, a number consisting of no higher order of figures than tens. Again, the square of tens can consist of no higher order than thousands and no lower than hundreds, for the square of 90 is 8100 and the square of 10 is 100; i. e. the square of 9 tens gives no higher figure than thousands, and the square of 1 ten gives no lower significant figure than hundreds.

In like manner we might show to what orders of figures the squares of hundreds, thousands, etc., would belong; also to what orders the cubes, biquadrates, etc. of units, tens, hundreds, etc., would belong.

312. Examples in Powers.

1. What is the 3d power of 5?

   Ans. $5^3 = 5 \times 5 \times 5 = 125$.
2. What is the 6th power of 6?
3. What is the product of 6^4 multiplied by 6^2?
4. What is the product of $3^5 \times 3^3$?
5. What is the 3d power of $\frac{6}{21}$? Ans. $\frac{8}{343}$.
6. What is the square of $3\frac{1}{4}$?
7. What is the cube of $\frac{7}{3}$?
8. What is the square of .25? Ans. .0625.
9. What is the cube of .006?
10. What is the 6th power of 1?
11. What is the quotient of $7^6 \div 7^4$? Ans. 49.
12. What is the quotient of $9^8 \div 9^5$?
13. What is the quotient of $10^{19} \div 10^{18}$?
14. How many figures are there in the cube of 99? Ans. 6.
15. How many figures are there in the cube of 40?
16. How many figures are there in the cube of 12?
17. How many figures are there in the 5th power of 99?
18. How many figures are there in the 5th power of 10?

A TABLE OF POWERS.

1st	1	2	3	4	5	6	7	8	9
2d	1	4	9	16	25	36	49	64	81
3d	1	8	27	64	125	216	343	512	729
4th	1	16	81	256	625	1296	2401	4096	6561
5th	1	32	243	1024	3125	7776	16807	32768	59049
6th	1	64	729	4096	15625	46656	117649	262144	531441
7th	1	128	2187	16384	78125	279936	823543	2097152	4782969
8th	1	256	6561	65536	390625	1679616	5764801	16777216	43046721
9th	1	512	19683	262144	1953125	10077696	40353607	134217728	387420489
10th	1	1024	59049	1048576	9765625	60466176	282475249	1073741824	3486784[illegible]1

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§ 37. EVOLUTION.

313. EVOLUTION is the reverse of *Involution.*

In Involution, the root is given and the power required.

In Evolution, the power is given and the root required.

314. A *root* of a number is one of the *equal* factors whose continued product is that number (94, b, Note 1).

The number of times the root is to be taken as a factor depends upon the *name* of the root.

The *square root* of a number is one of its *two* equal factors; the *cube root* is one of its *three* equal factors; etc.; thus, the *square root* of 64 is 8, for $8 \times 8 = 64$; the *cube root* of 64 is 4, for $4 \times 4 \times 4 = 64$; the *sixth root* of 64 is 2, for $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$.

315. Powers and roots are *correlative terms;* i. e. if one number is the *square* of another, then the latter is *necessarily* the *square root* of the former; thus, if 9 is the *square* of 3, then 3 *must be* the *square root* of 9.

316. There are two methods of *indicating* a root, — one by means of the *radical sign,* $\sqrt{\ }$, and the other by means of a *fractional index* (94, b, Notes 3 and 4).

The figure placed over the radical sign is the index of the root, and is always the same as the denominator of the fractional index. If no number is over the radical sign, 2 is understood.

317. EVOLUTION *or* EXTRACTING ROOTS *is the resolving of a quantity into as many equal factors as there are units in the index of the root.*

318. As we *involve* a number by *multiplying* its index by the number denoting the power (304), so we *evolve* a number by *dividing* its index by the number denoting the root; thus, the square root of 8^6 is $8^{6 \div 2} = 8^3$, for $8^3 \times 8^3 = 8^{3 \times 2} = 8^6$; the cube root of 6^{12} is $6^{12 \div 3} = 6^4$, for $6^4 \times 6^4 \times 6^4 = 6^{4 \times 3} = 6^{12}$; etc. Following out this principle, (which is universal,) we introduce *fractional indices;* thus, the square root of 4, i. e. 4^1, is $4^{\frac{1}{2}} = 2$; the cube root of 64, i. e. 64^1, is $64^{\frac{1}{3}} = 4$; etc.

319. In like manner, we may indicate a power and a root at the same time; thus, the square root of 4^3 is $4^{\frac{3}{2}}$. In this and all similar expressions, the numerator of the index indicates a power and the denominator indicates a root, and it is immaterial which is read first, the power or the root; *for any power of any root of any number is equal to the same root of the same power of the same number;* thus, $8^{\frac{2}{3}}$ is the square of the cube root of 8 or it is the cube root of the square of 8, and either result is 4.

320. We may also indicate a power and a root at the same time by means of an index and a radical sign; thus, $\sqrt[3]{8^5} = 32$, is the 5th power of the cube root of 8 or it is the cube root of the 5th power of 8; $\sqrt{16^3} = 64$ is the cube of the square root of 16 or it is the square root of the cube of 16; etc.

321. *All* numbers can be *involved* to *any* required power, but comparatively *few* can be *evolved.* There are but 9 *integral* numbers less than 100 that are perfect squares and but 4 that are perfect cubes.

Those numbers which can have their roots extracted are called *perfect* powers and their roots are *rational* numbers. Numbers

whose roots cannot be taken are called *imperfect* powers and their roots are *irrational*, *radical* or *surd* numbers; the term *radical*, however, for convenience, is applied to all quantities standing under a radical sign or fractional index, whether their roots can or cannot be taken.

A number may be a perfect power of one name or degree and an imperfect power of another; thus, 16 is a perfect square but an imperfect cube, whereas 27 is a perfect cube but an imperfect square; again, 64 is a perfect square, cube and sixth power.

322. Every root of 1 is **1.** There is no other number whose powers and roots are all alike.

The roots of a proper fraction are greater than the fraction and the roots of any number greater than unity are less than the number; thus, $\sqrt{\frac{4}{9}} = \frac{2}{3} > \frac{4}{9}$; $\sqrt[3]{\frac{27}{64}} = \frac{3}{4} > \frac{27}{64}$; but $\sqrt{\frac{64}{36}} = \frac{8}{6} < \frac{64}{36}$; $\sqrt[3]{125} = 5 < 125$; etc.

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§ 38. EXTRACTION OF THE SQUARE ROOT.

323. To EXTRACT THE SQUARE ROOT *of a number is to resolve it into two equal factors, i. e. to find a number which, multiplied into itself, will produce the given number.*

324. The *square* of a number always consists of *twice* as many figures as the *root*, or of *one less* than twice as many (310); conversely, then, there will be *one* figure in the *root* for *each two* figures in the *square;* and if there is an *odd* figure in the square, there will be yet another figure in the root for that odd figure in the square.

325. Again, the square of units can consist of no higher order of figures than *tens*, and the square of tens of no lower order than *hundreds* (311); ∴, if a number consists of 3 or 4 figures, its square root must consist of 2 figures, tens and units,

and we must look for the square of the tens in the 3d or in the 3d and 4th places of the power.

326. Let us now take an example and see if we can discover any principles to guide us in extracting the square root of a number.

Ex. 1. How large a square floor can be laid with 576 square feet of boards?

If we knew the length and breadth of a floor, we should find its area by multiplying the length by the breadth, (75), or, in this example, (since length and breadth are equal,) by multiplying the length by itself. *But we are now to reverse this process, and, knowing the area, to find the length of one side.*

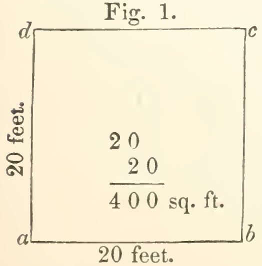
Since the number, 576, consists of *three* figures, its root will consist of *two* (324), *tens* and *units*, and the *square* of the tens must be found in the 5 (hundreds) (325).

OPERATION.

```
    5̇ 7 6̇ (2 4
    4
4 4) 1 7 6
     1 7 6
     -----
         0
```

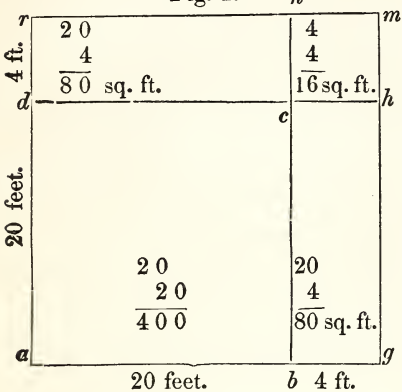
Now the square of 2 (tens) is 4 (hundreds) and the square of 3 (tens) is 9 (hundreds); and, as 5 (hundreds) is less than 9 (hundreds) there can be but 2 (tens) in the root. Let us now construct a square, Fig. 1, each side of which shall be 2 tens (= 20 feet) in length. The area of this square is $20 \times 20 = 400$ square feet, which, deducted from 576 feet, will leave 176 square feet to be used in enlarging the floor. To preserve the square form, this addition must be made upon 2 or 4 sides of the floor; for convenience we will make it upon 2 sides, as in Fig. 2. From the nature of the case, the 2 additions, *bm* and *cr*, are of a uniform breadth; and, if their *length* were known, we could determine their breadth by dividing their area, 176 feet, by their length (75, a). But we *do*

Fig. 1.



know the length of $bh + cr$, viz., twice the tens of the root $= 4$ (tens or 40 ft.), and this is sufficiently near to the whole length of the additions, to serve as a *trial divisor.* Now $176 \div 40$, or, what is the same in effect, $17 \div 4$, gives 4 ft. for the breadth of the addition, and this added to the *trial divisor*, 40, or annexed to the 4 (tens) will give 44, the whole length of $bm + cr$, the *true divisor.* And $44 \times 4 = 176$; i. e. the length of the addition multiplied by its breadth gives its area.

Fig. 2.



It will be seen that every foot of board is used and the floor is in a square form, each side of which is $20 + 4 = 24$ ft. long; $\therefore$ the problem is solved.

327. The same species of reasoning applies, however many figures there may be in the root. Hence,

To Extract the Square Root of a number,

RULE.—1. *Separate the given number into periods of two figures each, by placing a dot over units, hundreds, etc.*

2. *Find the greatest square in the left hand period and set its root at the right, in the place of a quotient in long division.*

3. *Subtract the square of this root-figure from the left hand period, and to the remainder, annex the next period for a dividend.*

4. *Double the root already found for a* TRIAL DIVISOR, *and, omitting the right hand figure of the dividend, divide and set the*

quotient as the next figure of the root and also at the right of the trial divisor, and so form the TRUE DIVISOR.

5. *Multiply the true divisor by this new figure of the root and subtract the product from the dividend.*

6. *To the remainder annex the next period for a new dividend, and, having doubled the part of the root already found for a trial divisor, proceed as before until all the periods have been employed.*

NOTE 1.—The left hand period may consist of but one figure.

NOTE 2.—The trial divisor being smaller than the true divisor, the quotient is frequently too large, *and a smaller number must be set in the root.* This usually occurs when the addition to the square, $a\,c$, is wide, and, consequently, the square, $h\,n$, large; or, in other words, when the trial divisor is much less than the true divisor.

328. PROOF.—*Add the four parts of the square together;* thus,

$$\begin{array}{rl} a\,c = & 400 \\ b\,h = & 80 \\ c\,r = & 80 \\ h\,n = & \underline{16} \\ a\,m = & 576\text{, the area of the square.} \end{array}$$

2D MODE OF PROOF.—*Square the root;* thus, $24 \times 24 = 576$, the area, as before.

Ex. 2. What is the square root of 67081?

$$\begin{array}{r|l} & \dot{6}\,7\,\dot{0}\,8\,\dot{1}\ (2\,5\,9\text{, Ans.} \\ & 4 \\ \hline 45) & 2\,7\,0 \\ & 2\,2\,5 \\ \hline 509) & 4\,5\,8\,1 \\ & 4\,5\,8\,1 \\ \hline & 0 \end{array}$$

In this example, the left hand period consists of but *one* figure. So, also, the trial divisor, 4, is contained in 27 six times; and the 2d remainder, 45, equals the divisor; still, the true root figure is but 5.

3. What is the square root of 42016324?

```
        . . . .
        42016324(6482, Ans.
        36
    124)601
        496
   1288)10563
        10304
  12962)25924
        25924
            0
```

4. What is the square root of 580644? Ans. 762.
5. $\sqrt{1679616}$ = how many? Ans. 1296.
6. What is the square root of 15625? Ans. 125.
7. What is the square root of 390625? Ans. 625.
8. What is the square root of 9765625? Ans. 3125.
9. What is the square root of 1073741824? Ans. 32768.
10. $119550669121^{\frac{1}{2}}$ = ? Ans. 345761.
11. What is the square root of 59048912180241? Ans. 7684329.
12. What is the square root of 16777216?

```
      . . . .
      16777216(4096, Ans.
      16
  809)7772
      7281
 8186)49116
      49116
          0
```

When a root-figure is 0, as in this example, we simply annex 0 to the trial divisor and bring down the next period to complete the new dividend.

13. What is the square root of 3486784401? Ans. 59049.
14. What is the square root of 41211436036? Ans. 203006.
15. What is the square root of 5764801?
16. What is the square root of 43046721?
17. What is the square root of 60466176?
18. What is the square root of 282475249?

19. What is the square root of 104.8576? Ans. 10.24.

```
           . . . .
          104.8576(10.24
          1
202)0485
      404
  2044)8176
        8176
           0
```

As a number consisting of an integer and a decimal is involved just as an integral number is involved, pointing off for decimals, as in Art. 156, so the root of a number, partly integral and partly decimal, is extracted precisely as though it were a whole number, taking care to place the first dot over *units* and pointing both to right and left. If the right-hand period is deficient, we annex a cipher: for this will complete the period but will not affect the value of the decimal (149).

There will be as many *integral* figures in the root as there are *periods of integral figures* in the power, and for each period of *decimals* in the power there will be a *decimal* figure in the root.

If the entire power is decimal, place the first dot over hundredths and point towards the right.

20. What is the square root of 747.4756? Ans. 27.34.

21. What is the square root of 4698? Ans. 68.541+.

```
            . .
          4698(68.541
          36
    128)1098
        1024
   1365)7400
         6825
  13704)57500
         54816
 137081)268400
         137081
         131319
```

If there is a remainder after employing all the periods in the given example, the operation may be continued at pleasure by annexing successive periods of ciphers, decimally; there will, however, in such examples, *always be a remainder;* for the right-hand figure of the dividend is *a cipher,* whereas the right-hand figure of the subtrahend is, *necessarily,* the right-hand figure of the square of some one of the nine digits, *the right-hand figure of the root and of the divisor being always alike.* Now, no one of the nine digits, squared,

will give a number ending with a cipher; ∴, the last figure of the dividend and of the subtrahend being unlike, *there must be a remainder.*

22. What is the square root of 19.876? Ans. 4.458+.
23. What is the square root of 176.94328?
24. What is the square root of 25.467?
25. What is the square root of 396.18475?
26. What is the square root of 872.94?
27. What is the square root of 187.946?
28. What is the square root of 49.87604?
29. What is the square root of $\frac{32}{50}$?

$$\sqrt{\tfrac{32}{50}} = \sqrt{\tfrac{16}{25}} = \tfrac{4}{5}, \text{ Ans.}$$

To extract the root of a vulgar fraction,

Reduce the fraction to its simplest form, and then take the root of the numerator and denominator separately; or, if either term of the fraction, when reduced, is an imperfect square, reduce the fraction to a decimal (158), *and then proceed as in the foregoing examples.*

30. What is the square root of $\frac{108}{147}$? Ans. $\frac{6}{7}$.
31. What is the square root of $\frac{144}{289}$? Ans. $\frac{12}{17}$.
32. What is the square root of $\frac{3}{4}$? Ans. .866+.
33. What is the square root of $930\frac{1}{4}$?

$$\sqrt{930\tfrac{1}{4}} = \sqrt{\tfrac{3721}{4}} = \tfrac{61}{2} = 30\tfrac{1}{2}, \text{ Ans.}$$

34. What is the square root of $\frac{1}{2} + \frac{3}{4} + \frac{7}{8} - \frac{9}{16}$?

$$\sqrt{\tfrac{1}{2} + \tfrac{3}{4} + \tfrac{7}{8} - \tfrac{9}{16}} = \sqrt{\tfrac{25}{16}} = \tfrac{5}{4} = 1\tfrac{1}{4}, \text{ Ans.}$$

35. What is the square root of $\frac{1}{50}$ of $\frac{7}{8} + \frac{1}{15}$ of $\frac{3}{4} - \frac{1}{200}$? Ans. $\frac{1}{4}$.

36. What is the square root of $\dfrac{3\frac{1}{4}}{8} + \dfrac{\frac{2}{5}}{\frac{4}{5}}$ of $\dfrac{3\frac{1}{9}}{\frac{7}{3}} + 5\frac{1}{2} + \frac{8}{9}$? 3.122 Ans. 3.12279+.

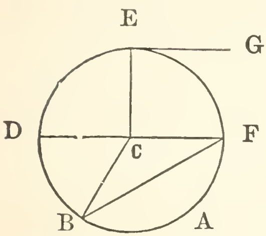
37. What is the square root of $\dfrac{\frac{3}{4}}{\frac{7}{9}}$ of $\dfrac{6\frac{1}{4}}{8} + \dfrac{3\frac{1}{4}}{5\frac{2}{3}}$ of $\frac{1}{2} - \frac{1}{3}$? ~~3.122~~

38. What is the square root of .376942784?

329. Application of the Square Root.

DEFINITIONS.

Fig. 1.



1. A Circle is a plane figure bounded by a curved line, all parts of the line being equally distant from a point within, called the center.

2. The *curve* which bounds the circle is called the *circumference.*

3. The *circumference* is divided into 360 equal parts, called *degrees* (81); ∴ 180° is semicircumference, 90° is quadrant, 60° is sextant, 45° is octant, etc.

4. Any portion of the circumference, e. g. BAF, is called an *arc.*

5. The straight line BF, which joins the extremities of any arc is called a *chord.*

6. Any chord DF which passes through the *center* of a circle. is called a *diameter.*

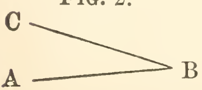
The *diameter* is longer than any other chord.

7. Any straight line, as CB, CD, CE, etc., drawn from the center to the circumference, is called a *radius* or *semi-diameter.*

8. Any straight line, as EG, which touches the circumference in one point E, and can touch it in no other point when the line is extended, is called a *tangent.*

The point E is called the *point of contact* or *point of tangency.*

Fig. 2.



9. When two lines meet, as in Fig. 2, they are said to form an *angle.*

The point B, where the lines meet, is called the *vertex* of the angle.

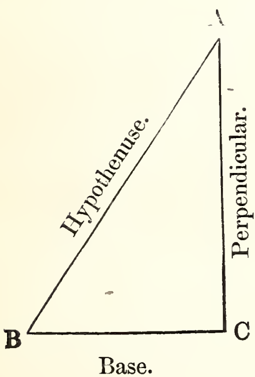
We call this the angle B; if, however, there are several angles at one point, as at C in Fig. 1, it is necessary to designate the angle by *three* letters; e. g. in Fig. 1, BCD or DCB, *the middle letter* ALWAYS *standing at the vertex*, and the other two letters at the other extremities of the lines which form the angle designated.

10. If a circle be drawn having its center at the vertex of an angle, the *arc*, included between the sides of the angle, is said to *measure* the angle; thus, if the arc BD, Fig. 1, is $\frac{1}{8}$ of the circumference, ($=45°$,) the angle BCD is an angle of 45°. If the arc is $\frac{1}{4}$ of the circumference, ($=90°$,) as DE in the angle DCE, or EF in ECF, the angle is one of 90° and is called a *right angle.*

The lines which form a *right angle* are *perpendicular to each other.*

An angle of less than 90°, as BCD, Fig. 1, is an *acute angle.* An angle of more than 90°, as BCF, Fig. 1, is an *obtuse angle.* Lines forming acute or obtuse angles are *oblique to each other.*

FIG. 3.



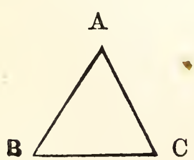
11. A TRIANGLE is a figure bounded by three straight lines.

12. A *right-angled triangle* has *one* of its angles a *right angle.*

The side opposite the right angle is called the *hypotheneuse;* the other two sides are the *base* and *perpendicular.*

The sum of the three angles of any plane triangle is equal to *two right angles*, or 180°.

FIG. 4.



13. An *equilateral triangle* has its three sides *equal to each other.*

Fig. 5.

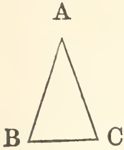
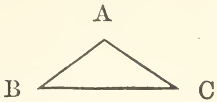


Fig. 6.

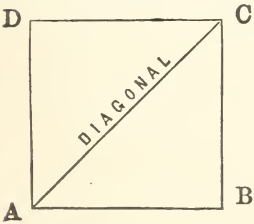


14. An *isosceles triangle* has *two* and *only two* of its sides equal.

Fig. 7.

15. A rectangle is a four sided figure, each of whose angles is a *right angle.*

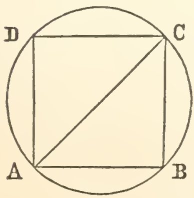
Fig. 8.



16. A Square is an *equilateral rectangle.*

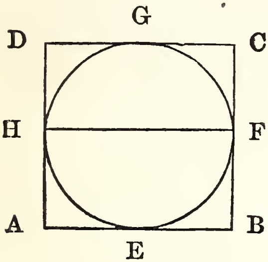
17. A *diagonal* is a straight line, as AC, joining the vertices of two opposite angles.

Fig. 9.



18. A square, or any other figure, having the vertex of each of its angles in the circumference of a circle is *inscribed* in that circle; and the circle is *circumscribed* about the figure.

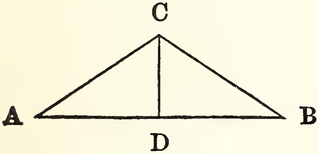
FIG. 10.



19. A square, or any other figure, having each of its sides tangent to a circle, is *circumscribed* about that circle; and the circle is *inscribed* in the figure.

330. By Geometry, the following propositions are easily demonstrated: —

FIG. 11.



1. If in a triangle, a line be drawn from the vertex of an angle included between equal sides, perpendicular to the third side, it will *bisect* that third side; i. e., it will divide the third side into *two equal parts.*

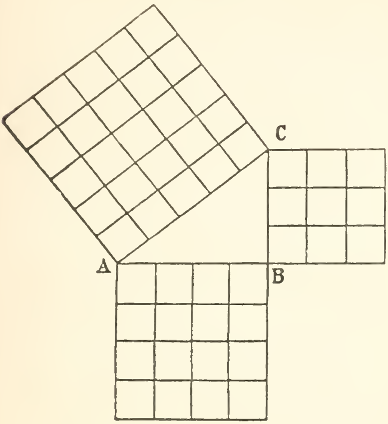
2. The diagonal of a square (Fig. 8.) divides the square into *two equal right angled triangles.*

3. The diameter of any circle is to its circumference in the ratio of 1 to 3.141592, nearly; hence the diameter multiplied by 3.141592 will give the circumference, and, conversely, the circumference divided by 3.141592 will give the diameter.

4. The area of a circle may be found by multiplying the square of its diameter by .785398, nearly, and, conversely, if the area is divided by .785398, the quotient will be the square of the diameter.

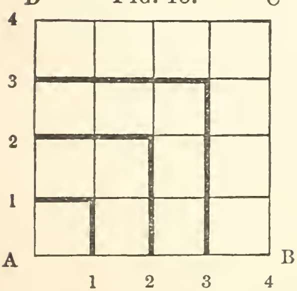
5. The areas of two circles are to each other as the squares of their radii, diameters or circumferences.

FIG. 12.



6. The square described on the hypothenuse of a right-angled triangle is equal to the sum of the squares described on the other two sides This will be seen by counting the small squares in the square of the hypothenuse and those in the squares of the other two sides. The square of either side adjacent to the right angle is equal to the square of the hypothenuse *minus* the square of the other side.

FIG. 13.



7. A square described on a line 1 foot long is only $\frac{1}{4}$ as great as a square described on a line 2 feet long, $\frac{1}{9}$ as great as that on a line 3 feet long, etc., etc.

8. If two angles of one triangle are respectively equal to two of another triangle, then the third angle of the one is equal to the third angle of the other and the two triangles are similar.

9. In similar triangles the sides of one are proportional to the homologous or corresponding sides of the other, and the areas are as the squares of those sides.

10. If one angle of a right-angled triangle is 30°, the side opposite that angle will be $\frac{1}{2}$ as long as the hypothenuse.

EXAMPLES.

1. A certain square field contains 10 acres of land; how many rods in length is one side of this field? Ans. 40.

2. A square field contains 20 acres; how many feet in the diagonal of this field? Ans. 1320.

3. A tree, broken off 21 feet from the ground and resting on the stump, touches the ground 28 feet from the stump; what is the length of the part broken off? Ans. 35 feet.

4. A fort 24 feet in hight, standing by the side of a stream, can be reached from the opposite side of the stream by a ladder that is 40 feet in length; what is the width of the stream?

Ans. 32ft.

5. A rope 100 feet long, attached to the top of a derrick and drawn perfectly straight, reaches the ground 80 feet from the derrick; how high is the derrick? Ans. 60ft.

6. A field in the form of a right-angled triangle contains $1\frac{1}{8}$ acres, and the base of the triangle is $4\frac{4}{9}$ times as long as the perpendicular; what will it cost to fence this field, at $66\frac{2}{3}$c. per rod? Ans. $60.

7. Two ships sail from the same port, one due east and the other due south, one at the rate of 8 miles and the other 10 miles per hour. Suppose the surface of the ocean to be plane, how far apart are the ships in 24 hours?

8. What is the side of a square equal in area to a circle 50ft. in diameter? Ans. 44.311+ft.

9. What is the diameter of a circular pond which shall contain 16 times as much area as one ten rods in diameter?

10. What is the diameter of a circle whose area is $\frac{1}{4}$ as great as that of a circle whose circumference is 62.83184ft.?

Ans. 10ft.

11. An army consists of 546121 men; how many shall be placed in rank and file to form them into a square?

12. Two rafters, each 35 feet long, meet at the ridge of a roof 15 feet above the attic floor; what is the width of the house?

Ans. 63.245+ft.

13. A certain room is 25ft. long, 20ft. wide and 12ft. high, how far from one lower corner to the opposite upper corner?

Ans. 34.19+ft.

14. A pipe $\frac{3}{4}$ of an inch in diameter will fill a cistern in 5 hours; in what time will a pipe $2\frac{1}{4}$ inches in diameter fill it?

Ans. $33\frac{1}{3}$m.

15. If a wire $\frac{1}{5}$ of an inch in diameter sustain a weight of 1800lbs., what weight will be sustained by a wire $\frac{1}{2}$ an inch in diameter, the strength of the wire varying as the area of the transverse section?

16. A circular island 44 feet in diameter has a canal of uniform width around it. At the center of this island stands a statue 11 feet tall, and a line extending from the top of the statue to the opposite bank of the canal is 61 feet long. What is the width of the canal, if the land upon the two sides is upon the same level? Ans. 38ft.

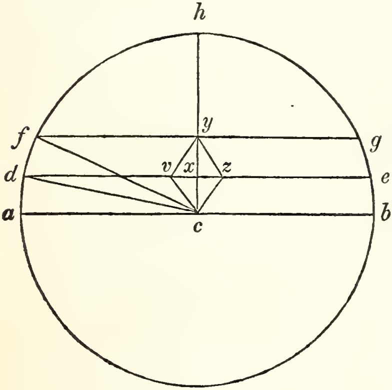
17. Four men buy a grindstone 3 feet in diameter; what length of radius shall each wear off successively so that each may wear off $\frac{1}{4}$ of the stone?

18. On a plane which makes an angle of 30° with the horizon, is a circle 300 feet in diameter. At the extremities of that diameter which extends from the lowest to the highest point of the circle, stand two vertical towers, the lower one being 300 feet and the upper one 250 feet tall. At a point directly between the two, and $\frac{1}{3}$ of the distance from the taller, stands a vertical column 10 feet in hight. What is the distance (1) from the top of this column to the top of each tower? (2) from the top of the column to the bottom of each tower? (3) from the top of one tower to the top of the other? (4) from the top of each tower to the bottom of the other? (5) from the top of each tower to the bottom of the column? (6) between the two towers horizontally? Ans. to (3), 278.388ft.; to (6), 259.807ft.

19. What would be the answer to these several questions if the towers and column stood on the horizontal diameter, the column 100 feet from the taller tower?

20. A young lady has a circular flower-plat, 14 feet in diameter. How many plants can be set upon it so that no two shall be within 10 inches of each other, and none within 4 inches of the circumference of the plat; a plant to occupy only a mathematical point? Ans. 241.

FIG. 14.



SUGGESTIONS.

The diameter is 14 feet = 168 inches long; but, as no plant is to stand within 4 inches of the circumference, $a\,b$, the part upon which plants may stand, is 160 inches long, and $a\,c$ is 80 inches. It is evident that 17 plants may be set on $a\,b$, one being at the center. Now, if $c\,v\,z$ is an equilateral triangle whose sides are 10 inches each, then $v\,x = 5$ inches (330, 1), and $c\,x = \sqrt{75}$; hence, in the right-angled triangle $c\,d\,x$, whose hypothenuse $c\,d$ is 80 inches, $d\,x$ can be found, from which $d\,v$ and the number of plants on $d\,e$ can be found. Again, $c\,y = 2 \times c\,x = 2 \times \sqrt{75}$, and the square of $c\,y = 4$ times the square of $c\,x$ (330, 7) $= 4 \times 75 = 300$; $\therefore$, in the right-angled triangle $c\,f\,y$, whose hypothenuse $c\,f$ is 80 inches, we can find $f\,y$ and $\therefore$ the number of plants on $f\,g$; etc.

21. A certain rectangular field containing 60 acres has its length to its breadth as 3 to 2; what are its length and breadth? Ans. 120 rods long and 80 rods wide.

22. What is the mean proportional between 2 and 380192? Ans. 872.

23. A boy, standing directly under a kite, is 225 feet from his companion, who holds the lower end of the string and has let out 375 feet; what is the hight of the kite? Ans. 300ft.

24. A ladder 25 feet long, set in a street, will reach a window 24 feet high upon one side of the street, and, without moving the foot, it will reach a window 16 feet high on the other side; what is the width of the street? Ans. 26.209ft.

25. A certain rectangular box, which is 10 inches long, contains 480 solid inches, and the depth of the box is to its breadth as 4 to 3; what is the distance from one upper corner to the diagonally opposite lower corner of the box?

Ans. $\sqrt{200} = 14.1+$in.

26. A number of men having contracted a joint debt of 40£ 16s. 9d., it was found that its payment required just as many pence from each man as there were men in the company; required the number of men in the company? Ans. 99.

27. What is the cost of fencing a rectangular field of 25 acres whose width is $\frac{2}{5}$ of its length, at 50cts. per rod?

Ans. $140.

28. I have 1200 apple-trees, which I wish to set out in a rectangular orchard, so that the number of trees in a row shall be 3 times the number of rows; the trees are to be 30 feet apart, and no tree is to stand within 10 feet of the fence. How large a field is required? Ans. 24a. 39rd. 4yd. 6ft. 36in.

29. What is the side of a square equivalent in area to a rectangular field, which is 121 rods long and 49 rods wide?

30. What will be the difference in the expense of fencing a circular 10-acre lot and one of the same area in a square form, the fence costing 50cts. per rod?

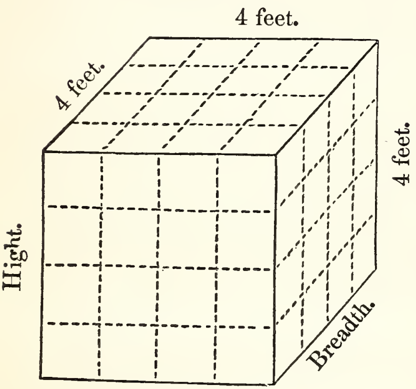
31. The acute angle at the base of a right-angled triangle is 30°, and the hypothenuse is 60 inches; how much less than 60 inches is the base?

32. Suppose 6 gallons of water flow through a pipe 1 inch in diameter in 1 minute, how many gallons would flow through a pipe 5 inches in diameter in 10 minutes, the streams moving with the same velocity?

33. I have a room whose length, breadth and hight is, each, 12 feet; what is the distance from one lower corner, through the center, to the opposite upper corner?

§ 39. EXTRACTION OF THE CUBE ROOT.

FIG. 1.



331. A CUBE is a solid (Fig. 1.), bounded by 6 equal square faces. Its contents are obtained (77) by multiplying its length, breadth and hight together, or (since these 3 dimensions are equal), by cubing either of the edges; conversely, if the contents are given, the length of one edge will be found by *extracting the cube root* of the number expressing the contents.

332. TO EXTRACT THE CUBE ROOT *of a number is to resolve it into 3 equal factors; i. e. to find a number which, multiplied into its square, will produce the given number.*

333. The cube of a number consists of three times as many figures as the root, or of **1** or **2** less than three times as many (310); conversely, there will be **1** figure in the root for each period of 3 figures in the cube, and an extra figure in the root if there are 1 or **2** figures over complete periods in the power; hence, to determine the number of figures in the root, we point off the number into periods of **3** figures, by placing a dot over units, thousands, etc.

Ex. 1. Suppose we have 74088 blocks of wood, each a cubic inch in size and form, how large a cubical pile can be formed by packing these blocks together?

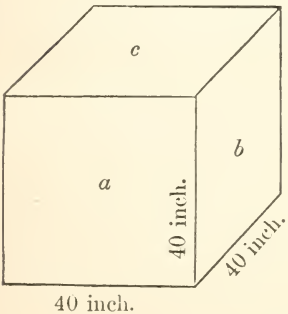
OPERATION.

Trial divisor,	4800	74088 (42
	240	64
	4	10088 Dividend.
True divisor,	5044	10088
		0

As there are *two* periods, the root must consist of *two* figures, tens and units; and, since the cube of tens cannot consist of any figure of a lower order than thousands (311), we seek the cube of the tens in the left hand period; the greatest cube in 74 is 64, whose root is 4. We place the root, 4, at the right of the number, and, having subtracted the cube, 64, from the left hand period, we annex the next period to the remainder, 10, making 10088 for a dividend.

Thus a cube is formed (Fig. 2.) whose edge is 40 inches and whose contents are 64000 solid inches, and there are 10088 blocks remaining, with which to enlarge the cubic pile already formed.

FIG. 2.



In enlarging this pile and preserving the cubic form, the additions must be made upon each of the 6 faces, or, more conveniently, equally upon any 3 adjacent faces, e. g. *a*, *b* and *c*, as in Fig. 3. What may be the thickness of the addition? By dividing the contents of a rectangular solid by the area of one face we obtain the thickness (77, a); now, the remaining 10088 solid inches are the contents, and the sum of the areas of the 3 square faces, *a*, *b* and *c*, is sufficiently near the area to be covered by the additions to form a *trial divisor;* for the 3 additions, *a*, *b* and *c* (Fig. 3.), are the same as one solid 40 inches wide, 3 times 40 inches long and of the thickness determined by trial. The area of these 3 faces is the square of 4 (tens), which is 16 (hundreds), multiplied

by 3, which gives 4800; i. e., to obtain a trial divisor, we square the root-figure and annex 00 (because the root-figure is *tens*) for the area of one face, and then multiply this area by 3. Dividing 10088 by 4800, we obtain the quotient 2, *for the thickness of the additions*, i. e. for the *unit* figure of the root. Having made these additions, as in Fig. 3, we see that the pile does not retain the cubic form, three corners, *m*, *m* and *m*, being vacant. Each of these corners is 40 inches long, 2 inches wide and 2 inches thick; i. e. the area covered to the depth of two inches by filling the vacant corners in Fig. 3, as seen in Fig. 4, is $2 \times 40 \times 3 = 240$ square inches; and still there is a vacant corner, *n*, *n*, *n*, (Fig. 4.) which is a cube

FIG. 3.

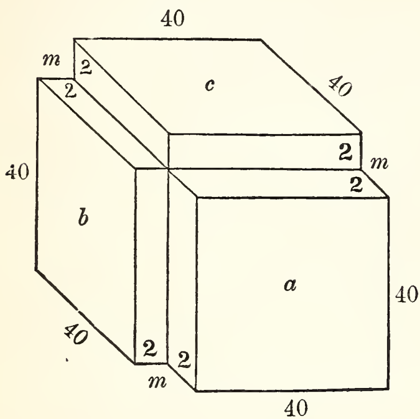
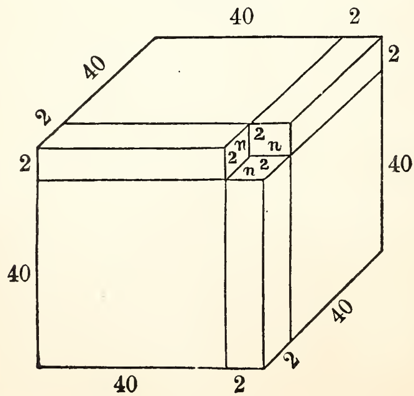
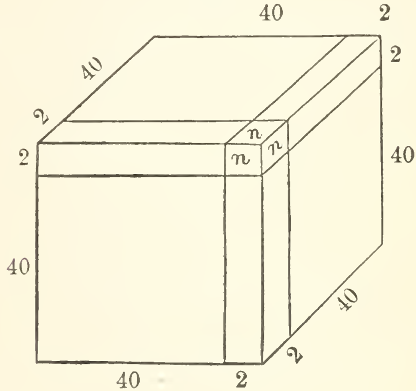


FIG. 4.



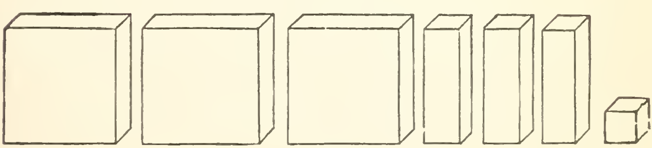
of 2 inches on each edge ; i. e. it is a solid 2 inches thick, (the *common* thickness of all the additions,) covering $2 \times 2 = 4$ square inches, as seen in Fig. 5.

FIG. 5.



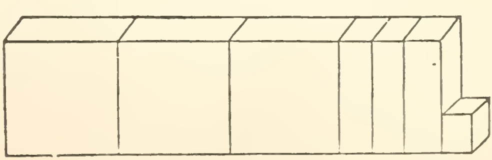
Now, if the several additions made in Figs. 3, 4 and 5, be spread out upon a plane, as in

FIG. 6,



or, in a consolidated form, as in

FIG. 7,



it will be readily seen that their collective solidity will be obtained by multiplying the entire area which they cover, $(40 \times 40 \times 3 + 40 \times 2 \times 3 + 2 \times 2 = 5044$ square inches,) by their *common* thickness, 2, which will give 10088 solid inches; ∴ a cube is formed, (Fig. 5.) whose edge is $40 + 2 = 42$ inches, and no blocks remain.

334. If there are more than two figures in the root, the same relations subsist, and the same reasoning applies. Hence,

To extract the Cube Root of a Number,

RULE. — 1. *Separate the number into periods of three figures each by setting a dot over units, thousands, etc.*

2. *Find by trial the greatest cube in the left-hand period, place its root as in square root, subtract the cube from the left-hand period and to the remainder annex the next period for a dividend.*

3. *Square the root figure, annex two ciphers and multiply this result by* 3 *for a* TRIAL DIVISOR; *divide the dividend by the trial divisor and set the quotient as the next figure of the root.*

4. *Multiply this root figure by the part of the root previously obtained, annex one cipher and multiply this result by* 3; *add the last product and the square of the last root figure to the trial divisor, and the* SUM *will be the* TRUE DIVISOR.

5. *Multiply the true divisor by the last root figure, subtract the product from the dividend and to the remainder annex the next period for a new dividend.*

6. *Find a new trial divisor, and proceed as before until all the periods have been employed.*

NOTE. — The notes in Art. 327, with slight modifications, are equally applicable here.

335. PROOF. — *Add the several parts; or, cube the root, and, if the result is like the given power, the work is probably right.*

Ex. 2. What is the cube root of 21024576?

		21024576(276, Ans.	
1st Trial Divisor $= 20^2 \times 3 =$	1200	8	
$20 \times 7 \times 3 =$	420		
$7^2 =$	49		
1st True Divisor $=$	1669	13024	1st Dividend.
2d Trial Divisor $= 270^2 \times 3 =$	218700	11683	
$270 \times 6 \times 3 =$	4860		
$6^2 =$	36		
2d True Divisor $=$	223596	1341576	2d Dividend.
		1341576	
		0	

The 1st trial divisor is contained 10 times in the dividend, yet the root figure is only 7. The true root figure can never exceed 9, *and must in all cases be found by trial.*

Squaring 20 gives the same result as squaring 2 and annexing 00, as directed in the rule, 3d paragraph.

3. What is the cube root of 67917312?

	6 7 9 1 7 3 1 2 (4 0 8, Ans.
4 8 0 0 0 0	6 4
9 6 0 0	
6 4	
4 8 9 6 6 4	3 9 1 7 3 1 2
	3 9 1 7 3 1 2
	0

In this example, the 1st trial divisor, 4800, is larger than the 1st dividend, 3917; ∴ we annex 0 to the root, 00 to the 1st trial divisor for the 2d trial divisor, and bring down the next period to complete a new dividend. The rule, followed literally, will give the same result.

4. What is the cube root of 491916472984? Ans. 7894.

5 What is the cube root of 27054036008? Ans. 3002.

6. What is the cube root of 2803221? Ans. 141.
7. What is the cube root of 1860867? Ans. 123.
8. What is the cube root of 95256152263? Ans. 4567.
9. What is the cube root of 3176523?
10. What is the cube root of 8024024008?
11. What is the cube root of 36926037?
12. What is the cube root of 10941048?
13. What is the cube root of 382657176?
14. What is the cube root of 75084686279296875?

Ans. 421875.

15. What is the cube root of 2? Ans. 1.2599+.

		$\dot{2}$ (1.2 5
1st Trial Divisor =	3 0 0	1
	6 0	
	4	
1st True Divisor =	3 6 4	1 0 0 0 1st Dividend.
2d Trial Divisor =	4 3 2 0 0	7 2 8
	1 8 0 0	
	2 5	
2d True Divisor =	4 5 0 2 5	2 7 2 0 0 0 2d Dividend.
		2 2 5 1 2 5
		4 6 8 7 5 0 0 0 3d Dividend.

Having obtained 1 for the first figure of the root, there is a remainder of 1, to which we annex a period of ciphers and proceed according to the *rule*.

The remaining figures of the root will be *decimals;* and, as in the Square Root, (328, Ex. 21.) so here, and for like reasons, *there will always be a remainder.*

16. What is the cube root of 3?
17. What is the cube root of 4?
18. What is the cube root of 7?
19. What is the cube root of 14?
20. What is the cube root of 19?
21. What is the cube root of 28?

22 What is the cube root of 3746.8792?

```
                               .   .   .   .
                               3746.879200(15.53+, Ans.
   1st Trial Divisor =     300|1
                           150|
                            25|-
                          ----
   1st True Divisor =      475|2746              1st Dividend.
  2d Trial Divisor =     67500|2375
                          2250|
                            25|----
                         -----
  2d True Divisor =      69775| 371879           2d Dividend.
 3d Trial Divisor =    7207500|348875
                         13950|
                             9|------
                       -------
 3d True Divisor =     7221459  23004200 3d Dividend.
                                21664377
                                --------
                                 1339823 4th Dividend.
```

In this example, the right-hand period was incomplete, and therefore it is completed by annexing two ciphers. The principle is the same as in Square Root (328, Ex. 19).

23. What is the cube root of 56.98742357?

24. $\sqrt[3]{84.604519}$ = how many? Ans. 4.39

25. What is the cube root of 73426.8741? Ans.

26. What is the cube root of 74.088?

27. What is the cube root of $\frac{54}{250}$?

$$\sqrt[3]{\tfrac{54}{250}} = \sqrt[3]{\tfrac{27}{125}} = \tfrac{3}{5}, \text{ Ans}$$

28. What is the cube root of $\frac{15379}{23625}$? Ans. $\frac{13}{15}$.

29. What is the cube root of $\frac{1}{5}$?

$$\sqrt[3]{\tfrac{1}{5}} = \sqrt[3]{.2} = .58+, \text{ Ans.}$$

When the exact root of a vulgar fraction cannot be extracted, it may be expedient first to reduce it to a decimal fraction, and then take the root.

30. What is the cube root of $\frac{24}{25}$?

31 What is the cube root of $3\frac{3}{8}$?

$$\sqrt[3]{3\tfrac{3}{8}} = \sqrt[3]{\tfrac{27}{8}} = \tfrac{3}{2} = 1\tfrac{1}{2}, \text{ Ans.}$$

32. What is the cube root of $4\frac{17}{27}$?
33. What is the cube root of $5\frac{9}{16}$?
34. What is the cube root of 5687943?
35. What is the cube root of 405.224?
36. What is the cube root of 4.68759?
37. What is the cube root of $\frac{64}{74088}$?

336. Application of the Cube Root.

1. Solids which are of the same form and have their like lines proportional are *similar;* thus, if one of two rectangular solids has its length 4 feet, its breadth 2 feet and its thickness 1 foot, and the other solid has its length 12 feet, its breadth 6 feet and its thickness 3 feet, *those two solids are similar.*

2. The like lines or parts of two similar solids or figures are called *homologous* lines or parts.

3. By Geometry it is easily proved that the solidities, i. e., the solid contents of all similar solids are to each other as the cubes of their homologous lines; thus, the solidities of two spheres are to each other as the cubes of their radii, as the cubes of their diameters, or as the cubes of their circumferences, etc., etc.; the solidities of cubes are to each other as the cubes of their edges; etc., etc.

Ex. 1. How many lead balls $\frac{1}{4}$ of an inch in diameter will be required to make a ball 1 inch in diameter? Ans. 64.

2. If a man dig a cubical cellar whose edge is 5 feet in one day, how long will it take him to dig a similar cellar whose edge is 25 feet? Ans. 125 days.

3. Suppose the diameter of the sun is 886144 miles and that of the earth 7912 miles, how many bodies like the earth will make one as large as the sun? Ans. 1404928.

4. If an iron ball 5 inches in diameter weighs 16lb., what is the weight of an iron ball 20 inches in diameter?

Ans. 1024lb.

5. If a globe of gold 1 inch in diameter is worth \$100, what is the diameter of a globe worth \$2700? Ans. 3 inches.

6. A, B, C and D own a conical sugar loaf which is 16 inches high and weighs 16lb.; what part of the hight shall each take off in the order A, B, C and D, so that each shall take 4lb.?

Ans. A, 10.079 in.; B, 2.620in.; C, 1.837in.; D, 1.464in.

7. A half-peck measure is $9\frac{1}{4}$ inches in diameter and 4 inches deep; what are the dimensions of a similar measure that will hold a bushel? Ans. $18\frac{1}{2}$ by 8 inches.

8. A rectangular bin, containing 327680 cubic inches, has its width, hight and length in the ratio of 1, 2 and 5; what are its dimensions? Ans. 32in. wide; 64in. high; 160in. long.

9. What is the edge of a cubical box whose solidity is equal to that of a bin whose length, breadth and hight are respectively 144, 36 and 9 inches? Ans. 36 inches.

10. Suppose 1000 bodies like the earth are required to make 1 like Saturn and that the diameter of Saturn is 79000 miles; what is the diameter of the earth? Ans. 7900 miles.

11. Four spheres have their solidities to each other in the ratio of the numbers 1, 2, 3 and 4; the diameter of the largest sphere is 5 inches. What is the radius of the smallest and what the successive increase of the radii of the 2d, 3d and 4th?

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§ 40. TO EXTRACT A ROOT OF ANY DEGREE.

337. Rule.—1. *Point off the given number into periods of as many figures each as there are units in the index of the required root, by placing a dot over units, etc.*

2. *Find by trial, or by the table of powers* (312), *the greatest power of the same name as the root in the left hand period, and place its root as the first figure of the required root.*

3. *Subtract the power from the first period and to the remainder annex the first figure of the next period, for a dividend.*

4. *For a trial divisor, involve the part of the root already found to a power whose index is one less than that of the required root and multiply this power by the index of the root.*

5. *Divide, and the quotient will be the second figure of the root or something greater.*

6. *Involve the part of the root found, to a power of the same name as the root, subtract the power from the first two periods, and to the remainder annex the first figure of the next period, for a new dividend.*

7. *Find a new trial divisor and proceed in a similar manner until the entire root is obtained.*

NOTE 1.—The left hand period may be incomplete. If, in pointing a decimal, the right hand period is incomplete, annex one or more ciphers.

NOTE 2.—Sections 4 and 5 of the rule *aid* in finding the successive root figures; still *each must be found by trial.*

NOTE 3.—The last involution in solving a question is, at the same time, a proof of the work.

NOTE 4.—This rule is founded in Algebra, and cannot be easily explained to pupils unacquainted with that science.

Ex. 1. What is the 4th root of 390625?

$$
\begin{array}{rl}
 & \dot{3}90\dot{6}25 \; (25, \text{Ans.} \\
2^4 = & 16 \\
\text{Trial Divisor } 2^3 \times 4 = 32\;) & 230 \text{ Dividend.} \\
25^4 = & 390625 \text{ Subtrahend.} \\
 & 0
\end{array}
$$

2. What is the 5th root of 1282388557824?

$$
\begin{array}{rl}
 & 12\dot{8}2388\dot{5}5782\dot{4} \; (264, \text{Ans.} \\
2^5 = & 32 \\
\text{1st Trial Divisor} = 2^4 \times 5 = 80\;) & 962, \text{ 1st Dividend.} \\
26^5 = & 11881376, \text{ 2d Subtrahend.} \\
\text{2d Trial Div.} = 26^4 \times 5 = 2284880\;) & 9425095, \text{ 2d Dividend.} \\
264^5 = & 1282388557824, \text{3d Subtrahend}
\end{array}
$$

3. What is the 11th root of 131621703842267136?

Ans. 36.

4. What is the 7th root of 265748499571034880?

Ans. 222.

5. What is the 4th root of 3808955502493696?

Ans. 7856.

6. What is the 8th root of 2304537769717568l?

Ans. 111.

7. What is the 6th root of 233217204680499310881000000?

Ans. 24810.

NOTE 5. — Such roots as the 4th, 6th, etc., i. e., those roots whose indices are composite numbers may be more easily found by taking a root of a root; thus, the $\sqrt[4]{625} = \sqrt{\sqrt{625}} = \sqrt{25} = 5$. Again, the $\sqrt[6]{64} = \sqrt[3]{\sqrt{64}} = \sqrt[3]{8} = 2$; or thus, $\sqrt[6]{64} = \sqrt{\sqrt[3]{64}} = \sqrt{4} = 2$; but roots whose exponents are prime numbers, as the 5th, 7th, 11th, etc., cannot be extracted in this way.

§ 41. ARITHMETICAL PROGRESSION.

338. Any series of numbers *increasing* or *decreasing* by *a common difference* is said to be in ARITHMETICAL PROGRESSION; thus,

2, 5, 8, 11, etc., is an ascending series, and
30, 25, 20, 15, etc., is a descending series.

339. The several numbers forming a series are called *terms;* the first and last terms, *extremes;* the others, *means.* The difference between any two successive terms is the *common difference.*

340. In Arithmetical Progression, *five particulars claim special attention:—*

1st. The first term.

2d. The last term.

3d. The common difference.
4th. The number of terms.
5th. The sum of all the terms.

341. These particulars are so related to each other that if any *three* of them are given the other *two* can be found.

342. *Twenty cases* may arise in Arithmetical Progression, but it will be sufficient to notice a few of the more important ones.

343. In an ascending series, let 3 be the first term and 4 the common difference;

Then,

$$3 = \text{1st term.}$$
$$3 + 4 = 7 = \text{2d term.}$$
$$3 + 4 + 4 = 3 + 2 \times 4 = 11 = \text{3d term.}$$
$$3 + 4 + 4 + 4 = 3 + 3 \times 4 = 15 = \text{4th term.}$$
$$3 + 4 + 4 + 4 + 4 = 3 + 4 \times 4 = 19 = \text{5th term.}$$
$$3 + 4 + 4 + 4 + 4 + 4 = 3 + 5 \times 4 = 23 = \text{6th term.}$$

etc. etc.

Again, in a descending series, let 30 be the first term and 3 the common difference;

Then,

$$30 = \text{1st term.}$$
$$30 - 3 = 27 = \text{2d term.}$$
$$30 - 3 - 3 = 30 - 2 \times 3 = 24 = \text{3d term.}$$
$$30 - 3 - 3 - 3 = 30 - 3 \times 3 = 21 = \text{4th term.}$$
$$30 - 3 - 3 - 3 - 3 = 30 - 4 \times 3 = 18 = \text{5th term.}$$

etc. etc.

Thus we see that, in an ascending series, the *second* term is found by adding the common difference *once* to the *first* term; the *third* term, by adding the common difference *twice* to the *first* term; and, *generally*, *any* term is found by adding the common difference *as many times* to the *first* term as there are terms *preceding* the one sought.

A similar explanation may be given when the series is descending. Hence,

344. Prob. 1.—The first term, common difference and number of terms being given, to find the last or any other designated term.

Rule.—*Multiply the common difference by the number of terms preceding the required term; add the product to the first term if the series is ascending, or subtract the product from the first term if the series is descending, and the sum or difference will be the term sought.*

Ex. 1. If the first term of an ascending series is 6, the common difference 3 and the number of terms 10, what is the last term? $6 + 9 \times 3 = 33$, Ans.

2. The first term of a descending series is 75 and the common difference 4; what is the 16th term?

$75 - 15 \times 4 = 15$, Ans.

3. The first term of an ascending series is 2 and the common difference is 5; what is the 21st term? Ans. 102.

4. The 1st term of a descending series is 500 and the common difference is 10; what is the 46th term? Ans. 50.

5. A triangular orchard has 3 trees in the first row, 5 in the second, 7 in the third, and so on in arithmetical progression; how many trees were there in the 15th row? how many in the 50th?

6. What will be the amount of $100 at simple interest for 30 years at 6 per cent. per annum? Ans. $280.

7. If a man on a journey travel $3\frac{1}{2}$ miles on the first day, 6 miles the second day, and so on in arithmetical progression, how far will he travel on the 20th day? Ans. 51 miles.

8. A boy bought 20 doves, paying 1 cent for the 1st, 3 cents for the 2d, and so on; what did he pay for the 20th?

Ans. 39 cents.

345. By inspecting the formation of the series in Art. 343, it will be seen that *the difference between the extremes is equal to the common difference multiplied by 1 less than the number of terms;* e. g. the difference beween the 1st and 6th terms in the

1st series, Art. 343 ($23 - 3 = 20$), is the sum of 5 *equal* additions; $\therefore$ this difference, divided by 5 ($20 \div 5 = 4$), will give *one* of those additions; i. e. the common difference. Hence,

346. Prob. 2.—The extremes and number of terms being given, to find the common difference,

Rule.—*Divide the difference of the extremes by the number of terms less one, and the quotient will be the common difference.*

Ex. 1. The extremes of an arithmetical series are 5 and 47, and the number of terms is 7; what is the common difference?

$47 - 5 = 42$, and $42 \div 6 = 7$, Ans.

2. The extremes are 27 and 148 and the number of terms is 12: what is the common difference? Ans. 11.

3. The amount of \$1 at simple interest for 25 years is \$2.50; what is the rate per cent.? Ans. 6.

4. A man has 12 sons whose ages form an arithmetical series; the youngest is 1 year old and the oldest 34; what is the difference of their ages? Ans. 3 years.

5. A body falling from rest in a vacuum descends $16\frac{1}{12}$ feet in the first second of time and $241\frac{1}{4}$ feet in the 8th second; the increments of velocity in successive seconds being equal, what is the increment in one second? Ans. $32\frac{1}{6}$ feet.

(a) This rule enables us to find any number of arithmetical means between two given quantities; for the number of terms in a series is two greater than the number of means; hence the common difference may be found and then the series is formed by adding the common difference once, twice, etc., to the 1st term.

Ex. 6. Find 6 arithmetical means between 3 and 38.

$38 - 3 = 35$ and $35 \div 7 = 5$, the common difference.

Ans. 8, 13, 18, 23, 28, 33.

7. Find 4 means between 2 and 37. Ans. 9, 16, 23, 30.

8. Find 7 means between 1 and 17.

Ans. 3, 5, 7, 9, 11, 13, 15.

9. Find 5 means between 27 and 3.

Ans. 23, 19, 15, 11, 7.

(b) If the same number of means be found between the successive terms of an arithmetical series, these means, together with the terms of the original series, will constitute a new arithmetical series.

Ex. 10. If 2 means be found between the successive terms of the series 1, 7, 13, 19, what will be the new series thus formed?

$$7 - 1 = 6 \text{ and } 6 \div 3 = 2 ; \therefore,$$

Ans. 1, 3, 5, 7, 9, 11, 13, 15, 17, 19.

11. Form an arithmetical series by inserting 3 means between the successive terms of the series, 2, 18, 34.

Ans. 2, 6, 10, 14, 18, 22, 26, 30, 34.

12. Form a series by inserting 4 means between the successive terms of the series 47, 32, 17, 2.

Ans. 47, 44, 41, 38, 35, 32, 29, 26, 23, 20, 17, 14, 11, 8, 5, 2.

347.—Again, it will be seen by inspecting Art. 343, that the *difference of the extremes* is the *sum* of the quantities added to or subtracted from the first term to obtain the last; and, as these quantities are equal, if their *sum* be divided by *one* of them, the quotient must be their number; i. e. *the quotient will be one less than the number of terms;* e. g. the difference between the 1st and 6th terms in the first series, Art. 343 ($23 - 3 = 20$), is the sum of a certain number of times 4; $\therefore$ this difference, divided by 4 ($20 \div 4 = 5$), will give the *number* of additions; i. e. *the number of terms less one.* Hence,

348. PROB. 3.—The extremes and common difference being given to find the number of terms,

RULE.—*Divide the difference of the extremes by the common difference, add 1 to the quotient and the sum will be the number of terms.*

Ex. 1. The extremes of an arithmetical series are 6 and 38 and the common difference is 4; what is the number of terms?

$$38 - 6 = 32 ; \; 32 \div 4 = 8 ; \text{ and } 8 + 1 = 9, \text{ Ans.}$$

2. The extremes of a series are 7 and 37; the common difference is 3. What is the number of terms? Ans. 11.

3. The ages of the scholars in a certain school are in arithmetical progression, the common difference of the series being 2 years; now, the youngest scholar is 5 years old and the oldest 35. What is the number of scholars? Ans. 16.

4. The extremes of a series being 8 and 29, and the common difference $4\frac{1}{5}$, what is the number of terms? Ans. 6.

5. A stone falling, descends $16\frac{1}{12}$ feet in the 1st second and $209\frac{1}{12}$ feet in the last second; the increments of velocity per second being $32\frac{1}{6}$ feet, how many seconds does it fall?

Ans. 7.

349. In an ascending series the second term is as much greater than the first as the last but one is less than the last; the third is as much greater than the first as the last but two is less than the last; etc.; ∴ the sum of the extremes is equal to the sum of any other two terms which are equally distant from the extremes; thus, in the series 1, 4, 7, 10, 13, 16, consisting of 6 terms, we have

$$\begin{array}{llllll} 1\text{st} + & 6\text{th} = & 2\text{d} + & 5\text{th} = & 3\text{d} + & 4\text{th} \\ 1 + & 16 = & 4 + & 13 = & 7 + & 10 = 17; \end{array}$$

and ∴ the sum of all the terms is $17 \times 3 = 51$.

Again, if the series consists of *an* ODD *number of terms*, say **7**; e. g. 2, 7, 12, 17, 22, 27, 32, then,

$$\begin{array}{llllll} 1\text{st} + & 7\text{th} = & 2\text{d} + & 6\text{th} = & 3\text{d} + & 5\text{th} \\ 2 + & 32 = & 7 + & 27 = & 12 + & 22 = 34, \end{array}$$

and there will be left the middle term, $17 = \frac{1}{2}$ of 34; i. e. the sum of the series is equal to $3\frac{1}{2}$ times 34 or 7 times $\frac{1}{2}$ of $34 = 119$. In a descending series, the reasoning is entirely similar. Hence,

350. PROB. 4.—The extremes and number of terms being given to find the sum of the series,

RULE 1.—*Multiply the sum of the extremes by half the number of terms;* or,

RULE 2.—*Multiply half the sum of the extremes by the number of terms and the product will be the sum of the series.*

NOTE.—One or the other of these rules can always be applied without *introducing* fractions into the calculations.

Ex. 1. The extremes of a series are 3 and 23 and the number of terms is 6; what is the sum of the series?

$3 + 23 = 26$; $6 \div 2 = 3$; and $26 \times 3 = 78$, Ans.

2. The extremes of a series are 5 and 47 and the number of terms is 15; what is the sum of the series?

$5 + 47 = 52$; $52 \div 2 = 26$; and $26 \times 15 = 390$, Ans.

3. The extremes of a series are 2 and 92 and the number of terms is 10; what is the sum of the series? Ans. 470.

4. The extremes of a series are 4 and 28 and the number of terms is 9; what is the sum of the series? Ans. 144.

5. The clocks of Venice strike from 1 to 24; how many strokes in 24 hours? Ans. 300.

6. How many strokes from a common clock in 24 hours?

7. Suppose a number of apples are placed in a straight line at the distance of one rod from each other for 10 miles, and that a basket is placed in the same line one rod from the first and nearest apple; how far must a person travel who goes from the basket to each apple separately and, returning, deposits it in the basket? Ans. 32030 miles and 2 rods.

8. The same conditions continuing as in the last example, except that the basket be 3 rods from the first apple, what distance must be traveled? Ans. 32070 miles and 6 rods.

9. By the laws of falling bodies, the distances which they fall in successive seconds of time constitute an arithmetical series, $16\frac{1}{12}$ feet in the 1st second, and $144\frac{3}{4}$ feet in the 5th second; how far will a body fall in 5 seconds? Ans. $402\frac{1}{12}$ feet.

351. PROB. 5.—The first term, common difference, and number of terms being given to find the sum of the series.

RULE.—*Find the last term by Prob.* 1, *and the sum of the series by Prob.* 4.

Ex. 1. The first term of an arithmetical series is 3, the common difference 4, and the number of terms 11; what is the sum of the series?

$4 \times 10 = 40$; $40 + 3 = 43$, the last term.

$3 + 43 = 46$; $46 \div 2 = 23$; and $23 \times 11 = 253$, Ans.

2. The first term is 5, common difference 8, and number of terms 21; what is the sum of the series? Ans. 1785.

3. The first term is $1\frac{1}{2}$, common difference $\frac{1}{2}$, and number of terms 41; what is the sum of the series? Ans. $471\frac{1}{2}$.

4. A falling body descends $16\frac{1}{12}$ feet in the first second of time, and in successive seconds of time the increments of velocity are $32\frac{1}{6}$ feet; how far will a body fall in 6 seconds?

Ans. 579 feet.

352. Since the sum of a series is found by multiplying the sum of the extremes by half the number of terms (350), so, conversely, if the sum of a series be divided by half the number of terms, *the quotient must be the sum of the extremes*, from which, if either extreme be subtracted, the remainder will be the other extreme. Hence,

353. Prob. 6.—The sum of the series, the number of terms, and either extreme being given, to find the other extreme.

Rule.—*Divide the sum of the series by half the number of terms; from the quotient subtract the given extreme, and the remainder will be the other extreme.*

Ex. 1. The sum of an arithmetical series is 57, the number of terms 6, and the least term 2; what is the greatest term?

$6 \div 2 = 3$; $57 \div 3 = 19$; and $19 - 2 = 17$, Ans.

2. The sum of a series is 196, number of terms 7, and least term 7; what is the greatest term? Ans. 49.

3. The sum of a series is 352, number of terms 11, and greatest term 57; what is the least term? Ans. 7.

4. A falling body descends $1029\frac{1}{3}$ feet in 8 seconds, in the 8th second it falls $241\frac{1}{4}$; how far does it fall in the 1st second?

Ans. $16\frac{1}{12}$ feet.

5. A gentleman owing 10 creditors $200, paid the 1st $2, and the others in arithmetical progression; what did he pay the last? Ans. $38.

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§ 42. GEOMETRICAL PROGRESSION.

354. Any series of numbers *increasing by a common multiplier* or *decreasing by a common divisor*, is said to be in GEOMETRICAL PROGRESSION; thus,

2, 6, 18, 54, etc., is an ascending series, and
64, 32, 16, 8, etc., is a descending series.

355. Here, as in *Arithmetical Progression*, the numbers forming the series are called *terms;* the first and last, *extremes;* the others, *means.*

The constant multiplier or divisor is the *ratio.* The ratio *may,* in every series, be considered a *multiplier, integral* when the series is *ascending* and *fractional* when it is *descending;* thus, in the 2d series above, the ratio is 2 if considered as a *divisor,* and $\frac{1}{2}$, as a *multiplier.*

356. Here, also, five particulars claim our attention:

1st. The first term.
2d. The last term.
3d. The ratio.
4th. The number of terms.
5th. The sum of all the terms.

357. Any *three* of these five particulars being given, the other *two* may be found.

358. *Twenty cases* may arise, but the investigation of several of them requires a knowledge of logarithms and the higher Algebraic equations, and, of the remaining cases, it will be sufficient for our purpose to present a few.

359. In an ascending series, let 3 be the first term, and 4 the ratio;

Then,

$$
\begin{aligned}
& 3 = \text{1st term.}\\
3 \times 4 = \quad & 12 = \text{2d term.}\\
3 \times 4 \times 4 = 3 \times 4^2 = \quad & 48 = \text{3d term.}\\
3 \times 4 \times 4 \times 4 = 3 \times 4^3 = \quad & 192 = \text{4th term.}\\
3 \times 4 \times 4 \times 4 \times 4 = 3 \times 4^4 = \quad & 768 = \text{5th term.}\\
3 \times 4 \times 4 \times 4 \times 4 \times 4 = 3 \times 4^5 = \quad & 3072 = \text{6th term.}\\
\text{etc.} \qquad & \text{etc.}
\end{aligned}
$$

Again, in a descending series, let 243 be the first term and $\frac{1}{3}$ the ratio;

Then,

$$
\begin{aligned}
& 243 = \text{1st term.}\\
243 \times \tfrac{1}{3} = \quad & 81 = \text{2d term.}\\
243 \times \tfrac{1}{3} \times \tfrac{1}{3} = 243 \times (\tfrac{1}{3})^2 = \quad & 27 = \text{3d term.}\\
243 \times \tfrac{1}{3} \times \tfrac{1}{3} \times \tfrac{1}{3} = 243 \times (\tfrac{1}{3})^3 = \quad & 9 = \text{4th term.}\\
243 \times \tfrac{1}{3} \times \tfrac{1}{3} \times \tfrac{1}{3} \times \tfrac{1}{3} = 243 \times (\tfrac{1}{3})^4 = \quad & 3 = \text{5th term.}\\
\text{etc.} \qquad & \text{etc.}
\end{aligned}
$$

In forming the above series we see that the *second* term is found by multiplying the *first* term by the *ratio;* the *third* term, by multiplying the *first* by the *square* of the ratio; the *fourth*, by multiplying the *first* by the *cube* of the ratio, and so on — *the index of the power of the ratio always being one less than the number of the term sought.* Hence,

360. Prob. 1.—The first term, ratio and number of terms being given, to find the last or any other assigned term.

Rule.—*Multiply the first term by that power of the ratio whose index is equal to the number of terms preceding the required term, and the product will be the term sought.*

Ex. 1. The first term of a geometrical series is 7, the ratio 3, and the number of terms 5; what is the last term?

$5 - 1 = 4$; $3^4 = 81$; and $81 \times 7 = 567$, Ans.

2. The first term of a series is 3, and the ratio 2; what is the ninth term?

$9 - 1 = 8$; $2^8 = 256$; and $256 \times 3 = 768$, Ans.

3. The first term is 64, and the ratio $\frac{1}{2}$; what is the tenth term? Ans. $\frac{1}{8}$.

4. A boy bought 17 oranges, agreeing to pay 1 mill for the 1st, 2 mills for the 2d, and so on in geometrical progression; what was the cost of the 17th orange? Ans. $65.536.

5. What is the amount of $1 at compound interest for 6 years at 6 per cent. per annum? Ans. $1.418519112256.

NOTE.—In Ex. 5, the 1st term is $1, the ratio is 1.06, and the number of terms 7.

6. Suppose Gen. Washington had put $100 to interest, Dec. 31, 1780, what, in justice, would be due his heirs, Dec. 31, 1900, allowing it to double every 12 years? Ans. $102400.

7. The estimated value of the estate of the Rosthchilds is now (1855) $40000000; what will be its value in 1975, allowing it to double once in 12 years? Ans. $40960000000.

8. Suppose a farmer to plant 1 kernel of corn and to harvest 1000 kernels, and suppose him to plant his entire crop from year to year and to harvest in the same ratio, what will be the value of his 10th year's crop if 1000 kernels make 1 pint, and he sells his corn at 1.12\frac{1}{2}$ per bushel?

Ans. $17578125000000000000000.

361. Since the last term is obtained (360) by multiplying the first term by that power of the ratio whose index is equal to the number of terms less one, so, conversely,

PROB. 2.—The extremes and number of terms being given, to find the ratio,

RULE.—*Divide the last term by the first, and the quotient will be that power of the ratio whose index is one less than the number of terms; the corresponding root of the quotient will therefore be the ratio.*

Ex. 1. The first term in a geometrical series is 3, the last term 192 and the number of terms 4; what is the ratio?

$$192 \div 3 = 64;\ 4 - 1 = 3;\ \text{and } \sqrt[3]{64} = 4, \text{ Ans.}$$

2. The first term is 160, the last term 5, and the number of terms 6; what is the ratio?

$5 \div 160 = \frac{1}{32}$; $6 - 1 = 5$; and $\sqrt[5]{\frac{1}{32}} = \frac{1}{2}$, Ans.

3. The extremes are 2 and 486, and the number of terms 6; what is the ratio? Ans. 3 or $\frac{1}{3}$.

(a) This rule enables us to find any number of geometrical means between two given numbers; for the number of terms in a series is two greater than the number of means: hence the ratio may be found, and then the series is formed by multiplying the first term by the *ratio*, by its *square*, its *cube*, etc.

4. Find 3 geometrical means between 2 and 512.

$512 \div 2 = 256$; $\sqrt[4]{256} = \sqrt{\sqrt{256}} = \sqrt{16} = 4$, ratio; $\therefore$ 8, 32, 128 are the means, and 2, 8, 32, 128, 256 = the series.

5. Find the series formed by 1 and 256, and 7 geometrical means. Ans. 1, 2, 4, 8, 16, 32, 64. 128, 256.

(b) If the same number of means be found between the successive terms of a geometrical series, these means, together with the terms of the original series, will form a new geometrical series.

6. If 3 means be found between the successive terms of the series 3, 768, 196608, what will be the new series thus formed?

Ans. 3, 12, 48, 192, 768, 3072, 12288, 49152. 196608.

7. Form a new series by inserting 2 means between the successive terms of the series 128, 16, 2, $\frac{1}{4}$, $\frac{1}{32}$.

Ans. 128, 64, 32, 16, 8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$.

362. Having a geometrical series given, e. g. 3, 12, 48, 192, 768, 3072, 12288, can we devise any short method for ascertaining the *sum* of all the terms?

Let us *multiply* each term *except the last* by the *ratio*, 4; thus,

	3, 12, 48, 192, 768, 3072, [12288], the given series.
Prod. by 4,	12, 48, 192, 768, 3072, 12288;

and we shall evidently form a *new* series like the *old*, except the

first term of the *old* is not found in the *new*. Now, if the *old* series *except the last term* be *subtracted* from the *new*, the *remainder will be the difference of the extremes in the old series*, the other terms in the two series canceling each other; the remainder will also be 3 times the *sum* of all the terms except the last in the old series; for *once* a series from 4 *times* a series *must* leave 3 *times* the series; ∴ $\frac{1}{3}$ of this remainder *plus the last term must be the sum of all the terms in the old series;* but 3 is the ratio less 1.

A similar explanation is always applicable. Hence,

363. Prob. 3. — The extremes and ratio being given, to find the sum of the series,

Rule. — *Divide the difference of the extremes by the ratio less* 1, *and to the quotient add the greater extreme.*

Ex. 1. The extremes are 2 and 20000, and the ratio 10; what is the sum of the series?

$20000 - 2 = 19998$; $10 - 1 = 9$; $19998 \div 9 = 2222$; and $2222 + 20000 = 22222$, Ans.

2. The extremes are 7 and 45927, and the ratio 3; what is the sum of the series? Ans. 68887.

3. The extremes of a series are 5 and 5120, and the ratio 4; what is the sum? Ans. 6825.

364. Prob. 4. — The first term, ratio and number of terms being given, to find the sum of the series,

Rule. — *Find the last term by Prob.* 1, *and the sum of the series by Prob.* 3.

Ex. 1. The first term is 5, the ratio 3 and the number of terms 9; what is the sum of the series?

$3^8 \times 5 = 32805$, last term; $(32805 - 5) \div 2 = 16400$, difference of extremes divided by the ratio less one; $16400 + 32805 = 49205$, sum of the series.

2. The first term is 7, ratio 6 and the number of terms 13; what is the sum of the series? Ans. 1828471621.

3. A lady being married on the first day of January, her father gave her \$1, promising to give her \$10 on the first of February, and so on in geometrical series on the first of the remaining months of the year; to what sum did her dowry amount? Ans. \$111111111111.

4. Had the ratio been 5 instead of 10 in the above example, what would have been the lady's dowry? Ans. \$61035156.

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§ 43. ANNUITIES.

365. AN ANNUITY is, properly, a sum of money payable *annually;* but the term is also applied to sums payable *monthly, quarterly, semi-annually, biennially, or at any regular intervals.*

Rents, salaries, pensions, etc., are annuities.

366. An annuity payable at a definite number of times is called *a certain annuity;* if payable periodically for an indefinite time, e. g. during the life of an individual, it is *a contingent* or *life annuity;* if payable at regular intervals forever, e. g. the interest of a school fund, it is *a perpetual annuity* or *a perpetuity.*

367. An annuity already commenced, or to commence immediately, is said to be *in possession;* if not to commence until a definite time has elapsed, or until the occurrence of a specified event, it is *in reversion;* if not paid when it becomes due, it is *in arrears.*

368. The *amount* of an annuity in arrears is the sum of all the installments which are due and unpaid, together with all the interest which has arisen on such installments.

369. The right to some species of annuities may be bought; in such cases, the purchase money is the *present worth* of the annuity.

370. The amount of annuities in arrears, at simple and at compound interest, may be found in various ways.

371. PROB. 1.—To find the amount of an annuity at simple interest.

EX. 1. What is the amount of an annuity of \$100 per year in arrears for 4 years, on simple interest at 6 per cent. per annum?

1ST METHOD.—The 4th installment, becoming due to-day, is worth just \$100: the 3d installment, having been due 1 year, amounts to \$106; so the 2d and 1st installments, having been due 2 and 3 years, respectively, amount to \$112 and \$118; ∴ \$100 + \$106 + \$112 + \$118 = \$436, the sum sought; but these numbers constitute an *arithmetical series*, of which the first term is the annuity, the common difference is the interest of the annuity at the given per cent. for the time between two successive payments, and the number of terms is the number of payments; ∴ we find the amount of the annuity by the rule in Art. 351.

2D METHOD.—As the several installments are on interest for 1, 2 and 3 years, it is plain that the entire interest is equal to the interest of \$100 for 1 year multiplied by (1 + 2 + 3); i. e. the entire interest = \$6 × 6 = \$36, and this added to the sum of the 4 installments, viz., \$400, gives \$436, as in the 1st method. Hence,

RULE.—*Find the sum of the natural series of numbers,* 1, 2, 3, *etc., up to the number of installments, less one, by Art.* 351; *multiply the interest of one installment for one interval of time, by this sum, and the product will be the entire interest; add the entire interest to the sum of all the installments and the whole sum will be the amount required.*

EX. 2. If an annual pension of \$500 be in arrears for 6 years, what will it amount to at 6 per cent. simple interest?

Ans. \$3450.

3. What is the amount of a salary of \$225 quarterly, in arrears for 4 years, at 6 per cent. per annum, simple interest?

Ans. \$4005.

4. What is the amount of an annual salary of $6000, in arrears for 8 years, at 7 per cent. simple interest.

Ans. $59760.

5. If a semi-annual rent of $350 be in arrears for 3 years and 6 months, what will it amount to at 8 per cent. simple interest?

6. The interest on a certain sum is $600 per annum; if this interest remains unpaid for 3 years, what, in justice, would be due the creditor, money being worth 10 per cent.?

7. What, money being worth 6 per cent.?

372. PROB. 2.—To find the amount of an annuity in arrears at compound interest,

Ex. 1. What is the amount of $1 annuity per annum, in arrears for 4 years, at 6 per cent. compound interest?

The 4th installment, becoming due to-day, is worth just $1; the 3d, having been due 1 year, is worth $1.06; so the 2d and 1st installments, having been due for 2 and 3 years respectively, amount, at compound interest, to $1.1236 and $1.191016; ∴ $1. + $1.06 + $1.1236 + $1.191016 = $4.374616, the sum sought; but these numbers constitute a *geometrical series*, of which the first term is the annuity, the ratio is the amount of $1 at the given rate for the time between two successive payments, and the number of terms is the number of payments; ∴ we find the amount of the annuity by the rule in Art. 364.

Ex. 2. If an annual pension of $500 be in arrears for 6 years, what will it amount to at 6 per cent. compound interest?

Ans. $3487.6592688.

REMARK.—In the several Problems in Annuities $1 may be considered the annuity, and having proceeded with $1 according to the rule, the product of the result multiplied by the true annuity will give the true result. Hence the utility of the following

TABLE,

Showing the amount of the annuity of \$1, £1, *etc., at* 4, 5, 6 *and* 7 *per cent. compound interest, for any number of years not exceeding* 20.

Years.	4 per cent.	5 per cent.	6 per cent.	7 per cent.
1	1.000000	1.000000	1.000000	1.000000
2	2.040000	2.050000	2.060000	2.070000
3	3.121600	3.152500	3.183600	3.214900
4	4.246464	4.310125	4.374616	4.439943
5	5.416323	5.525631	5.637093	5.750739
6	6.632975	6.801913	6.975319	7.153291
7	7.898294	8.142008	8.393838	8.654021
8	9.214226	9.549109	9.897468	10.259803
9	10.582795	11.026564	11.491316	11.977989
10	12.006107	12.577893	13.180795	13.816448
11	13.486351	14.206787	14.971643	15.783599
12	15.025805	15.917127	16.869941	17.888451
13	16.626838	17.712983	18.882138	20.140643
14	18.291911	19.598632	21.015066	22.550488
15	20.023588	21.578564	23.275970	25.129022
16	21.824531	23.657492	25.672528	27.888054
17	23.697512	25.840366	28.212880	30.840217
18	25.645413	28.132385	30.905653	33.999032
19	27.671229	30.539004	33.759992	37.378965
20	29.778079	33.065954	36.785591	40.995492

Ex. 3. What is the amount of an annual pension of \$900 in arrears for 17 years, at 7 per cent. compound interest?

Ans. \$27756.1953.

4. What is the amount of an annual salary of \$1000 which has been in arrears 20 years, at 5 per cent. compound interest?

Ans. \$33065.954.

5. What is the amount of an annual rent of \$150, in arrears for 12 years, at 6 per cent. compound interest?

Ans. \$2530.49115.

6. What is the amount of an annuity of \$300 per annum, in arrears for 15 years, at 4 per cent. compound interest?

Ans. \$6007.0764.

7. What is the amount of a quarterly salary of $225 in arrears for 4 years, allowing $1\frac{1}{2}$ per cent. interest per quarter, and compounding the interest quarterly? Ans. $4034.78+.

8. What is the amount of a semi-annual dividend of $500 in arrears for 4 years, allowing 3 per cent. interest for 6 months time, and compounding the interest semi-annually?

9. What is the amount of a biennial salary of $10000 in arrears for 8 years, allowing 12 per cent. interest for 2 years, compounding the interest biennially?

10. What is the amount of an annual rent of $300, in arrears for 19 years?

373. Prob. 3. — To find the present worth of a *certain* annuity at compound interest,*

Rule 1. — *Find the present worth of each installment, and the sum of these will be the present worth of the annuity; or,*

Rule 2. — *Find the amount of the annuity as though it were in arrears, and then discount this amount for the time to elapse before the last installment becomes due.*

Note. — These two rules will give the same result, but the 2d is the easier to apply.

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* To find the present worth of a *certain* annuity, discounting at *simple interest,* some authors have given this rule: — Find the present worth of each installment separately, and the sum of these will be the present worth of the annuity. Others find the amount of the annuity as though it were in arrears, and then discount this amonnt for the time to elapse before the last installment is due.

These rules will give different results, but the difference is unimportant; for to purchase an annuity by either of these rules would be in the highest degree absurd, since the present worth of an annuity for about 25 years at 6 per cent. by the 2d rule, or 30 years by the 1st, would be so great that its *annual interest would be more than the annual installment of the annuity;* e. g. the present worth of an annuity of $100 for 25 years, found by the 2d rule, is $1720. Now the *loan* of $1720 will entitle the lender to $103.20 interest annually, *forever,* and the principal would still be due; whereas the purchase of the annuity of $100 for 25 years by the payment of $1720, its present worth, will only secure the payment of $100 annually for 25 years, and neither installment nor the refunding of purchase money subsequently.

Ex. 1. What is the present worth of \$100 annuity, payable annually for 4 years?

OPERATION BY RULE 1.

$\frac{\$100}{1.06} = \$94.33\frac{102}{106}$ = present worth of 1st installment.
$\frac{\$100}{1.06^2} = \$88.99\frac{10836}{106^2}$ = present worth of 2d installment.
$\frac{\$100}{1.06^3} = \$83.96\frac{229664}{106^3}$ = present worth of 3d installment.
$\frac{\$100}{1.06^4} = \$79.20\frac{11824768 0}{106^4}$ = present worth of 4th installment.
$\$346.51\frac{7085904}{106^4}$ = present worth of annuity, Ans.

OPERATION BY RULE 2.

$\$100 \times 1.06^3 = \$119.1016$ = last term of series (360); $(\$119.1016 - \$100) \div (1.06 - 1) + \$119.1016 = \$437.4616$ = amount of \$100 annuity in arrears for 4 years (363); $\$437.4616 \div 1.06^4 = \$346.51\frac{7085904}{106^4}$, Ans. as before.

These operations may be much abridged by using the following

TABLE,

Showing the present worth of the annuity of \$1, £1, *etc., at* 4, 5, 6 *and* 7 *per cent., for any number of years not exceeding* 20.

Years.	4 per cent.	5 per cent.	6 per cent.	7 per cent.
1	.961538	.952381	.943396	.934589
2	1.886095	1.859410	1.833393	1.808018
3	2.775091	2.723248	2.673012	2.624316
4	3.629895	3.545950	3.465106	3.387211
5	4.451822	4.329477	4.212364	4.100197
6	5.242137	5.075692	4.917324	4.766546
7	6.002055	5.786373	5.582381	5.389289
8	6.732745	6.463213	6.209794	5.971299
9	7.435332	7.107822	6.801692	6.515232
10	8.110896	7.721735	7.360087	7.023582
11	8.760477	8.306414	7.886875	7.498676
12	9.385074	8.863252	8.383844	7.942686
13	9.985648	9.393573	8.852683	8.357651
14	10.563117	9.898641	9.294984	8.745468
15	11.118382	10.379658	9.712249	9.107914
16	11.652290	10.837770	10.105895	9.446652
17	12.165664	11.274066	10.477260	9.763223
18	12.659292	11.689587	10.827603	10.059087
19	13.133935	12.085321	11.158116	10.335595
20	13.590322	12.462210	11.469921	10.594014

Ex. 2. What is the present worth of an annuity of $60 per annum to continue 20 years, at 6 per cent. compound interest?

The present worth of $1 by the table is $11.469921,

$\therefore$ $11.469921 $\times$ 60 = $688.19526, Ans.

3. What is the present worth of an annuity of $175 per annum to continue 15 years, at 7 per cent. compound interest? Ans. $1593.884+.

4. What is the present worth of an annual pension of $150 for 12 years at 5 per cent.? Ans. $1329.4878.

5. A young man buys a farm for $2000, which he agrees to pay in 16 equal annual installments, the first in 1 year from the time of purchase. Allowing 6 per cent., what ready money will pay the debt? Ans. $1263.236+.

6. What is the present worth of a semi-annual salary of $500, to continue 8 years, allowing 4 per cent. interest for the time between two successive payments? Ans. $5826.145.

374. Prob. 4.—To find the present worth of a perpetuity.

The present worth of a perpetuity is, evidently, a sum whose interest for the interval between two successive payments is equal to one installment; now, interest is found by multiplying the principal by the rate per cent.; $\therefore$, conversely, the principal equals the interest divided by the rate per cent. Hence,

Rule.—*Divide the installment by the rate per cent. and the quotient will be the present worth of the perpetuity.*

Ex. 1. What is the present worth of a perpetuity of $60 per annum at 6 per cent.? $60 ÷ .06 = $1000, Ans.

2. What is the value of a perpetuity of $1200 per annum at 6 per cent.? Ans. $20000.

3. What is the present worth of a perpetuity of $900 per annum at 3 per cent? Ans. $30000.

375. Prob. 5.—To find the present worth of an annuity certain, in reversion.

Rule 1.—*Find the value of the annuity if entered on immediately and then discount that value for the time in reversion.*

Note.—A like rule will give the value of a perpetuity in reversion.

Rule 2.—*Find the present worth of a like annuity for the time in reversion, also for the whole time from the present till the last installment is due, and the difference of these will be the present worth of the annuity in reversion.*

Ex. 1. What is the present worth of \$500 annuity to commence in 3 years and continue 4 years, at 6 per cent.

BY RULE 1.

Present worth for 4 years = \$1732.553.

\$1732.553 ÷ 1.191016 = \$1454.684+, **Ans.**

BY RULE 2.

Present worth for 7 years = \$2791.1905.
Present worth for 3 years = \$1336.506.

Difference = \$1454.684+, Ans.

2. What is the value of a perpetuity of \$1000 to commence in 2 years, discounting at 6 per cent.? Ans. \$14833.274.

3. What is the present worth of an annual pension of \$300 to commence in 8 years and continue 12 years, discounting at 4 per cent.? Ans. \$2057.273.

376. Prob. 6.—To find an annuity, its present worth being given,

Rule.—*Divide the given present worth by the present worth of* \$1 *annuity for the given rate and time.*

Ex. 1. The present worth of an annuity for 3 years is \$500, what is the annuity?

\$500 ÷ 2.673012 = \$187.055 —, Ans.

2. The present worth of an annual rent for 10 years is \$6000; what is the rent, discounting at 5 per cent.? Ans. \$777.027.

377. Prob. 7.—To find an annuity, its amount being given,

Rule.—*Divide the given amount by the amount of* \$1 *annuity for the given rate and time.*

Ex. 1. The amount of an annuity for 4 years is \$600; what is the annuity, discounting at 6 per cent.?

$$\$600 \div 4.374616 = \$137.155 —, \text{ Ans.}$$

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§ 44. PERMUTATIONS, ARRANGEMENTS, AND COMBINATIONS.

PERMUTATIONS.

378. When several things are placed in a line in every possible order of succession, so that each shall enter every result, and enter it but once, they are said to be *permuted*, and *each order of succession* is called a *permutation;* thus, the single letter, a, can have but 1 position, i. e., it cannot stand either before or after itself; the 2 letters, a and b, furnish the 2 permutations, $\left\{\begin{matrix} a\,b \\ b\,a \end{matrix}\right\}$ the number of which is expressed by the product of $1 \times 2 = 2$; and if a 3d letter, c, be introduced, we have $\left\{\begin{matrix} c\,a\,b, & c\,b\,a \\ a\,c\,b, & b\,c\,a \\ a\,b\,c, & b\,a\,c \end{matrix}\right\}$; i. e., the new letter, c, may stand 1st, 2d, or 3d in each of the 2 permutations of a and b; hence the number of permutations of 3 things is expressed by the product, $1 \times 2 \times 3 = 6$. If a 4th letter, d, be taken, it may stand as 1st, 2d, 3d, or 4th, in each of the 6 permutations of a, b and c, and, of course, furnish 4 times $6 = 1 \times 2 \times 3 \times 4 = 24$ permutations.

By the above, it is evident that the No. of permutations

Of 1 thing		$= 1$
Of 2 things $=$	$1 \times 2 =$	2
Of 3 things $=$	$1 \times 2 \times 3 =$	6
Of 4 things $=$	$1 \times 2 \times 3 \times 4 =$	24
Of 5 things $=$	$1 \times 2 \times 3 \times 4 \times 5 =$	120

and so on to any extent. Hence,

379. PROB. 1.—To find the number of permutations of any given number of things,

RULE.—*Form the series of numbers,* 1, 2, 3, 4, *etc., up to the number of things to be permuted, and their continued product will be the number of permutations.*

Ex. 1. How many different integral numbers may be expressed by writing the 9 significant digits in succession, each figure to be taken once, and but once, in each number?

Ans. $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 = 362880$.

2. Suppose 20 books, standing on a shelf, be removed without noticing their order, what is the probability that the first trial to replace them as before will be successful?

Ans. As 1 to 2432902008176640000.

3. A family consists of father, mother, five sons, and five daughters; in how many different orders of succession may they arrange themselves around the dinner table?

Ans. 479001600.

4. The solar spectrum consists of 7 colors—red, orange, yellow, green, blue, indigo, and violet; in how many different orders may these colors be arranged? Ans. 5040.

ARRANGEMENTS.

380. If from any number of things a smaller number be selected in such a manner that all the groups shall consist of the same number of things, and each group be different from every other, in, at least, one of the things of which it is composed, these selections are called *combinations;* and if these combinations be permuted the results are called *arrangements;* thus, *ab*, *ac*, *ad*, *bc*, *bd*, *cd* are *combinations* of 4 letters in sets of 2 and 2, and *ab*, *ba*, *ac*, *ca*, etc., are *arrangements* of the same 4 letters in sets of 2 and 2.

It will be noticed that although *ab* and *ba are* 2 *arrangements*, yet they *are not* 2 *combinations*, for *ab* and *ba* are composed, *not of different, but of the* SAME *letters.*

381. If the 4 letters, a, b, c and d, be arranged in sets of 1 and 1, we have 4 arrangements, viz., a, b, c and d; and for *each* arrangement there are 3 *reserved letters — the* 3 *which are not in the set;* for the 1st arrangement, a, the reserved letters are b, c, and d; for the 2d, b, the reserved letters are a, c and d, etc., etc.

Now if we wish to arrange the 4 letters in sets of 2 and 2, we take *each* of the 4 arrangements in sets of 1 and 1 and annex to it each of the 3 reserved letters, thus,

$$\left\{\begin{matrix} ab, & ba, & ca, & da, \\ ac, & bc, & cb, & db, \\ ad, & bd, & cd, & dc, \end{matrix}\right\}$$

giving 4 times 3, i. e., $4 \times 3 = 12$ arrangements of 4 letters in sets of 2 and 2.

Again, if we would arrange them in sets of 3 and 3, we have but to annex *each* of the reserved letters to *each* of the 12 arrangements in sets of 2 and 2 and we have

$$\left\{\begin{matrix} abc, & acb, & adb, & bac, & bca, & bda, & cab, & cba, & cda, & dab, & dba, & dca, \\ abd, & acd, & adc, & bad, & bcd, & bdc, & cad, & cbd, & cdb, & dac, & dbc, & dcb. \end{matrix}\right\}$$

The *law* of arrangements is evident:—if we arrange any number of things in sets of 1 and 1, there will be as many arrangements as there are things from which to select; if in sets of 2 and 2, the number of arrangements will be equal to the number in sets of 1 and 1 multiplied by the number of reserved letters, i. e. the number from which we select, minus one; and so on in sets of 3 and 3, 4 and 4, etc. Hence,

382. PROB. 2.—To determine the number of arrangements that can be made of any number of things taken in sets of 1 and 1, 2 and 2, 3 and 3, etc., etc.

RULE.—*Form a series of numbers, beginning with the number of things from which we are to select, and decreasing by* 1 *until the number of terms is equal to the number of things to be taken at a time, and the continued product of these terms will be the number of arrangements.*

Ex. 1. How many integral numbers can be expressed, each composed of any 5 of the 9 significant figures?

Ans. $9 \times 8 \times 7 \times 6 \times 5 = 15120$.

2. How many arrangements can be made of 6 scholars out of a class of 16 scholars? Ans. 5765760.

3. How many arrangements of 6 letters, selected from the 26 of the English alphabet may be made? Ans. 165765600.

383. If the number of things in a set is the same as the number of things from which the selection is to be made, the question becomes one of *mere permutation;* e. g., what number of arrangements can be made of 4 books, taken in sets of 4 and 4?

Ans. $4 \times 3 \times 2 \times 1 = 24 =$ No. permutations of 4 things.

Combinations.

384. If *every possible combination* of any number of things in sets of 2 and 2, 3 and 3, etc., be permuted, these permutations *must be all the possible arrangements* of the same number of things in sets of 2 and 2, 3 and 3, etc.; e. g. if each of the 6 combinations of 4 things in sets of 2 and 2 be permuted, we obtain 12 arrangements,—all the possible arrangements of 4 things in sets of 2 and 2; i. e. the number of combinations of 4 things in sets of 2 and 2 multiplied by the number of permutations of 2 things gives the number of arrangements of 4 things in sets of 2 and 2; or, to abbreviate this and put it in the form of an equation, let A stand for the number of arrangements of 4 things in sets of 2 and 2; let P stand for the number of permutations of 2 things; and let C stand for the number of combinations of 4 things in sets of 2 and 2, and we have $C \times P = A$, and, dividing each member of this equation by P, we have $C = \frac{A}{P}$. This formula, reduced in the example under consideration, gives $\frac{4 \times 3}{1 \times 2} = 6$ combinations of 4 things in sets of 2 each.

A like explanation can be given to every example. Hence,

385. Prob. 3.—To find the number of combinations of any number of things in sets of 2 and 2, 3 and 3, etc.

Rule.—*Form a series of numbers, as in the rule for arrangements* (382), *for a dividend, and a series, as in the rule for permutations* (379), 1, 2, 3, *etc., up to the number of things to be combined at a time, for a divisor, and the quotient will be the number of combinations sought.*

Ex. 1. How many different companies of 7 men each may be selected from 21 men?

$$\frac{21 \times 20 \times 19 \times 18 \times 17 \times 16 \times 15}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7} = 116280, \text{ Ans.}$$

2. How many combinations of 6 letters, selected from the English alphabet, can be made? Ans. 230230.

3. How many different combinations of 3 colors can be made out of the 7 prismatic colors? Ans. 35.

4. The graduating class in a literary institution consists of 50 members, of whom 33 are to be selected for public speakers; how many different selections can be made?

Ans. 9847379391150.

5. Chemists describe 56 different elements in nature; now, if one particle in each element will combine with one particle in each of the other elements, how many combinations may be so formed? Ans. 1540.

6. A butcher bought 10 sheep out of a flock of 20, agreeing to pay as many cents for each of the 10 sheep as the owner could make different selections of 10 sheep from the flock; what did the sheep cost him? Ans. $18475.60.

386. If the number of things to be combined is the same as the number of things from which the selection is to be made, there can be but 1 combination, for the factors of the dividend and divisor will be the same, and, consequently, cancel each other; thus, e. g., how many combinations can be made of things selected from 6?

$$\text{Ans. } \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4 \times 5 \times 6} = 1.$$

387. Just as many combinations of 7 and 7 as of 2 and 2 may be made of 9 things; just as many of 6 and 6 as of 3 and 3, etc.; and, generally, just as many combinations of any number of things selected from a larger number can be made as of the remaining number after the smaller is taken from the larger; for, after as many factors at the beginning of the dividend and divisor each are considered as are in the smaller set, the remaining factors of the dividend and divisor cancel each other; thus, had the 1st example in Art. 385, read, — How many different companies of 14 men each may be selected from 21 men? — our formula would have been

$$\frac{21\times20\times19\times18\times17\times16\times15\,|\times14\times13\times12\times11\times10\times9\times8}{1\times2\times3\times4\times5\times6\times7\,|\times8\times9\times10\times11\times12\times13\times14}=116280,$$

in which all the factors of the numerator after 15 are like the factors of the denominator after 7 taken in an inverted order.

388. If the continued product of the terms of a descending arithmetical series, consisting of any number of integral terms whose common difference is 1, be divided by the continued product of the series 1, 2, 3, 4, etc., up to the number of terms whose product forms the dividend, *the quotient will be a whole number;* for this quotient will be the number of combinations of as many things in a set as are represented by the largest factor in the divisor, selected from the number represented by the largest factor of the dividend, *and this number of combinations is* NECESSARILY *a whole number;* thus,

$$\frac{79\times78\times77\times76}{1\times2\times3\times4}=1502501, \textit{ a whole number.}$$

§ 45. MISCELLANEOUS EXAMPLES.

389. Ex. 1. The sum of 3 numbers is 99; the 2d of these numbers is $3\frac{1}{2}$ times the 1st, and the 3d is $2\frac{1}{4}$ times the 2d. What are the numbers? Ans. 8, 28 and 63.

2. What is the interest on $356.50 from May 12, 1856, t July 4, 1858; at $7\frac{1}{2}$ per cent.?

3. Multiply $\frac{7}{13}$ of $7\frac{4}{11}$ by $\dfrac{5\frac{4}{7}}{2\frac{5}{11}}$. Ans. 9.

4. What number, multiplied by $\frac{1}{2}$ of itself, will produce $4\frac{1}{2}$? Ans. 3.

5. What number, multiplied by $\frac{1}{3}$ of itself, will produce $8\frac{1}{3}$? Ans. 5.

6. What number, multiplied by $2\frac{1}{5}$ times itself, will produce 55?

7. The hind wheel of a carriage is $10\frac{1}{2}$ feet, and the fore wheel 9 feet in circumference; how many revolutions will each make in running from Andover to Boston, $20\frac{1}{2}$ miles?

8. The salary of the President of the United States is $25000 per annum; what sum can he expend daily, and yet save $26950 in one term of office? Ans. $50.

9. A merchant has 14lb. sugar worth 10c., 14lb. worth 12c. and 28lb. worth 15c., which he wishes to mix with two other kinds worth 18c. and 23c. so as to make a mixture worth 19c. per lb.; how many pounds of each of the two latter kinds may he take? Ans. 56lb. at 18c. and 98lb. at 23c.

10. Hiero, King of Syracuse, ordered his jeweler to make him a crown of gold, weighing 63 ounces. The artist attempted a fraud by substituting a portion of silver; but the king, suspecting fraud, requested Archimedes to examine it. Archimedes, putting it into water, found that it displaced 8.2245 cubic inches of water; and, having found that an inch of gold weighs 10.36 ounces, and an inch of silver 5.85 ounces, he discovered what part of the king's gold had been purloined. It is required to repeat the process. Ans. 28.803+ ounces.

11. A rectangular piece of land, containing 30 acres, has its length to its breath as 3 to 1; what are its length and breadth?

Ans. 120 and 40 rods.

12. A gentleman, in disposing of his property, willed to his wife $\frac{1}{3}$ and to his son $\frac{2}{3}$ of his estate, if, of children, he left only a son; and to his wife $\frac{2}{3}$ and to his daughter $\frac{1}{3}$, if he left only a daughter. Now, at his decease, he left both a son and a daughter, in consequence of which his widow received \$3200 less than if he had left only a daughter. What would she have received if he had left only a son? Ans. \$2800.

13. Suppose all the conditions in Ex. 12 to remain unchanged, except that the gentleman shall leave a son and 2 daughters, what will be the answer?

14. A gentleman, dying, left his estate of 30983£ 14s. 6d. as follows, viz., for benevolent objects 8567£ 12s. 3d., to his widow 3567£, to each of his 3 daughters $\frac{1}{3}$ of $\frac{133338}{502643}$ of the remainder, and to each of his 5 sons, $\frac{1}{5}$ of what then remained; what was the share of a son? of a daughter?

15. Two men in Boston hire a carriage for \$20 to go to Fitchburg, 50 miles distant, and return, with the privilege of taking in 3 more persons; having gone 20 miles, they take in A; at Fitchburg they take in B, and when within 15 miles of the city they take in C. How much shall each man pay?

$$\text{Ans.} \begin{cases} \text{1st man,} & \$6.35. \\ \text{2d man,} & \$ \\ \text{A} & \$ \\ \text{B} & \$2.35. \\ \text{C} & \$ \end{cases}$$

16. A general, arranging his army in a square battalion, found that he had 116 men remaining; but, increasing the rank and file by one soldier, he wanted 129 men to make up the square. Of how many men did his army consist?

Ans. 15000.

17. If a pipe 6 inches in diameter will discharge a certain quantity of water in 4 hours, in what time will 3 four-inch pipes discharge twice the quantity. Ans. 6 hours.

18. A colonel, forming his regiment into a hollow square, found that, when he arranged the men 3 deep on each side of the square, he had 114 men left; but when he arranged them 4 deep, he wanted 114 men to complete the arrangement. Of how many men did his regiment consist? Ans. 750, or 846.

19. How shall I mark a package of wrought collars which cost me $4 each so that I may fall 20 per cent. from the marked price and yet make 25 per cent. on the purchase price?

Ans. $6.25.

20. What shall I ask per pair for gloves which cost $9.60 per dozen that I may discount $33\frac{1}{3}$ per cent. from the asking price and yet gain 25 per cent. on the cost?

21. What are the prime factors of 2800?

22. What are all the integral factors of 2800?

23. What is the greatest common measure of 1027 and 1781?

24. Bought of J. P. & Co. as follows:—

July 1, 1857,	on 60	days'	credit,	a bill	of	$200.
" 15,	" " 90	"	"	"		400.
Aug. 5,	" " 80	"	"	"		400.

Also, sold to J. P. & Co.:—

July 22, 1857,	on 2	months'	credit,	a bill of	$500.
Aug. 29,	" " 3	"	"	"	400.

When shall I pay the balance of the debt?

25. Sold to D. S. E. as follows:—

Jan. 8, 1857,	on 6	months,	10 acres of land at	$150,	$1500.
Feb. 28,	" " 3	"	5 tons of hay at	20,	100.

Also bought of him:—

March 4, 1857,	on 30	days,	6 horses at	$150,	$900.
" 26,	" " 40	"	8 cows at	25,	200.

When shall he pay me the balance of his debt?

26. What is the amount of $356, at 6 per cent. compound interest, from July 21, 1836, to July 31, 1857?

27. How many feet of boards are required to cover a house that is 40 feet long, 30 feet wide and 20 feet high to the top of

the plates, the ridge being 12 feet above the plates and the roof projecting 1 foot horizontally over the plates, no account being made of doors, windows, thickness of boards, etc. Ans. 4784.

28. How many bricks, whose dimensions are 2, 4 and 8 inches, are required to build the walls of a house 50 feet long, 32 feet wide and 20 feet high, the walls being 1 foot thick?

29. How many bricks would be required for the above mentioned house, if the walls were $1\frac{1}{2}$ feet thick? Ans. 127980.

30. What is the square root of 9 times the square of 16?

31. What is the square root of the square root of $\frac{9}{25}$ of the square of $\frac{12}{125}$? Ans. $\frac{6}{25}$.

32. The radii of two circles having a common center are 4 and 8 inches; what is the area of the circular ring included between the two circumferences? Ans. 150.796416sq. in.

REMARK.—The answer to Ex. 32 may be found without ascertaining the area of either circle. How?

33. A tree 32 feet tall, growing vertically upon a horizontal plane, is broken off so that the top reaches the ground 16 feet from the stub, the part broken off turning upon the top of the stub as upon a hinge; what is the hight of the stub?

Ans. 12 feet.

34. A gentleman being asked the time, replied that $\frac{3}{4}$ of the time past from noon was equal to $\frac{1}{11}$ of the time to midnight, what was the time?

35. Suppose a boy can count distinctly 180 per minute, how long will it take him to count one quadrillion by the French method of numeration? how long by the English method?

36. Received a quantity of goods from Liverpool, with instructions to sell them and invest the proceeds in cotton, after deducting a commission of $1\frac{1}{2}$ per cent. on the sales of the goods and 1 per cent. on the purchase of the cotton. Sold the goods at an advance of 5 per cent. on the invoice price and received \$12600; what was the invoice price and what sum was invested in cotton? Ans. Invoice, \$12000; invested, \$12288.11$\frac{89}{101}$.

37. How many different companies of 7 men each can be selected from 18 men? how many companies of 11 men each?

38. In a certain house there are 46 windows and 12 panes in each window. B buys this house, agreeing to pay 1 cent for the first pane of glass, 2 cents for the second, and so on, in geometrical progression, for all the panes; what is the price of the house?

39. What would have been the price of the above-mentioned house, had the series been in Arithmetical Progression?

40. Paid 3 debts successively, each of which took half of all the money I had before paying it and 50 cents more, and then had only $50 remaining; how much had I at first?

Ans. $407.

41. A cistern has a receiving and a discharging pipe. If the cistern be empty and both pipes open, it will be filled in 12 hours; whereas if the discharging pipe were closed, the cistern would be filled in 9 hours. In what time would the discharging pipe empty the full cistern, if the receiving pipe were closed?

42. What is the square of the cube root of $\frac{1}{5}$ of $\frac{3\frac{1}{4}}{5\frac{1}{2}}$ and $\frac{3}{2}$ of $\frac{2}{16\frac{446}{529}}$?

Ans. $\frac{4}{9}$.

43. What is the cube of the square root of $\frac{3}{8}$ of $\frac{5\frac{3}{5}}{4\frac{2}{3}} - \frac{1}{3}$ of $\frac{7\frac{1}{2}}{450}$?

Ans. $\frac{8}{27}$.

44. How many different integral numbers can be expressed by means of the ten Arabic digits, each digit being used once, and only once, in each number?

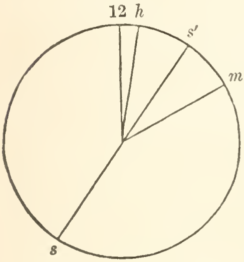
45. For what sum must a policy be taken out to secure an adventure of $10000 from New York to Liverpool, at 4 per cent.; thence to Havre at 1 per cent.; thence to San Francisco at 10 per cent.; and thence to New York at 8 per cent.?

46. If a certain number be increased by 5, and the sum diminished by 11, and the remainder multiplied by 4, and the

product divided by 3, the quotient will be 36; what is the number? Ans. 33.

47. A watch has an hour, a minute and a second hand, turning upon the same center-staff. At 12 o'clock the three hands are together. How long will it be (1) before the second-hand will be equally distant from the other two? (2) before the minute-hand will be equidistant from the other two? (3) before the hour-hand will be equidistant from the other two?

Fig. 1.



Analysis of (1)—Let h, m and s be the positions of the hour, minute and second-hands, severally, at the required time. Then since the minute-hand goes 12 times as fast as the hour-hand, and the second-hand 60 times as fast as the minute-hand, we shall have the distance

	From	12	to h =	1 space.
	"	12	to m =	12 spaces.
	"	12	to s =	720 "
∴	"	m	to s =	708 "
and	"	s	to h =	708 "
and	"	s	to 12 =	707 "
and	"	12 round	to 12 =	1427 "

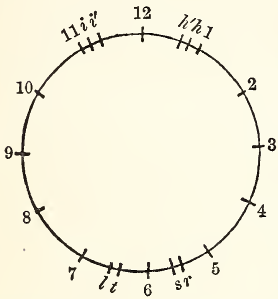
Hence the proportion:—As the distance round the clock is to the distance the hour-hand has moved, so is the number of seconds while the hour-hand is going round the clock to the number of seconds required; i. e.,

$$\text{As } 1427 : 1 :: 43200 \text{ sec.} : 30\tfrac{390}{1427} \text{ sec., Ans.}$$

Remark.—In a similar manner we may determine when the second-hand shall have the position s' represented in Fig. 1; also the questions (2) and (3), and all similar examples respecting the positions of the hands of a watch may be analyzed.

48. At what time between 12 and 1 o'clock will the hour and minute hands of a watch make equal acute angles with the line extending from the center-staff to 12? Ans. $55\frac{5}{13}$ *m.* past 12.

FIG. 2.



ANALYSIS.—At half-past 12 the minute-hand is at 6 and the hour-hand is at h', half way from 12 to 1. Now, if the hour-hand would stand still at h' while the minute-hand moved forward to i', half-way from 11 to 12, $27\frac{1}{2}$ minutes from the point 6, the hands would have the required positions; but, while the minute-hand is advancing, the hour-hand goes from h' to h; $\therefore$ the minute-hand must stop at i, as much short of i' as h is in advance of h'; i. e., the hour and minute hands together move over the space represented by $27\frac{1}{2}$ minutes on the dial; but the hour and minute hands together move over 13 spaces while the minute-hand alone moves over 12 spaces; hence the proportion: $13 : 12 :: 27\frac{1}{2}$ *m.* : $25\frac{5}{13}$ *m.*, the number of minutes beyond half-past 12 when the hands will have the required positions.

49. At what time between 5 and 6 o'clock do the hour and minute hands make equal acute angles with the line from 12 to 6?

50. At what time between 2 and 3 o'clock do the hour and minute hands point in opposite directions?

51. At a certain time between 8 and 9 o'clock the minute-hand was between 9 and 10. Within an hour afterwards the hour and minute hands had changed places. What was the 1st mentioned time?

52. An intelligent farmer, having a good pair of oxen fully shod, offered to sell them if any one would pay 1 cent for the first shoe, 2 cents for the second, and so on in geometrical progression for all the shoes. Having found a ready purchaser, it is required to find the price of the oxen. Ans. \$655.35.

53. The ignorant purchaser of the oxen mentioned in the above example, being astonished at the price, soon negotiated the sale of his horse, a noble animal, on the same terms. What did he receive for his horse? Ans. 15 cents.

54. Mr. Ignoramus, being chagrined by the bargains mentioned in Ex. 52 and 53, and resolving to retrieve his fortune by the purchase of a splendid mansion, offered the owner 1 cent for the 1st door, 2 cents for the 2d, 4 for the 3d, and so on. The house having 65 doors, what did it cost?

Ans. $368934881474191032.31.

55. The purchaser of the above-named house, being again astonished above measure, called in legal advice. The lawyer being a shrewd, practical man, counseled his client to say to the inexorable creditor:—"It is but reasonable, sir, that your legal demands be satisfied; please be seated till I can count the money, and you shall have your pay." Now, suppose the debtor can count $1 per second, 10 hours a day and 300 days in a year, how long must the creditor wait?

Ans. 34160637173 yr. 6 m. +.

56. What would be the simple interest on the above-named sum for the time required to count it?

57. What the amount at compound interest, allowing it to double once in 12 years?

SUPPLEMENT.

§ 46. MISCELLANY.

390. Arithmetical operations, bv the Roman Notation, are very cumbersome, and this may be both cause and consequence of there having been few, if any, Roman mathematicians of eminence.

391. The directions usually given for the *manner* of performing certain operations are merely for *convenience ;* thus, in Subtraction we are directed to write the subtrahend *under* the minuend; but one versed in figures will subtract as readily when the subtrahend is *over* the minuend or *elsewhere,* and it is frequently more convenient than to follow the rule.

Ex. Take	35769	
From	842076	In the proof, the upper number and remainder are added together.
Remainder,	806307	
Proof,	842076	

392. In Multiplication, the result will be the same, whichever figure of the multiplier is used first; still, there is usually no gain in departing from the common course, *but a decided loss in the lack of system.* The following example illustrates these points: —

```
Multiply      3742          3742          3742
By            4293          4293          4293
             -----         -----         -----
             33678         11226         11226
          14968         14968           33678
              11226        33678          7484
             7484          7484         14968
Product, 16064406 = 16064406 = 16064406
```

393. In Long Division, there may be a real gain in writing the divisor at the *right* instead of at the *left* of the dividend; for the work will be more *compact*, and *the divisor and quotient will have the usual relative position of factors in multiplication;* thus,

Dividend, 92250	(375 Divisor.	108£ 7s. 9d.(13	
750	(246 Quotient.	104	(8£ 6s. 9d.
1725	2250	4£	
1500	1500	20	
2250	750	87s.	
2250	92250 Proof.	78	
0		9s.	
		12	
		117s.	
		117	
		0	

Contractions in Multiplication.

394. To multiply by 25,

Rule. — *Annex* 00 *to the multiplicand (or move the decimal point two places towards the right) and divide the result by* 4.

Why? Because 25 is $\frac{1}{4}$ of 100, and $\therefore$ we wish to obtain $\frac{1}{4}$ of 100 times the multiplicand; thus,

Multiply 796 by 25. Also 7.96 by 25.

4) 7 9 6 0 0	4) 7 9 6.
1 9 9 0 0, Ans.	1 9 9., Ans.

395. To multiply by $33\frac{1}{3}$,

Rule. — *Annex* 00 *(or move, etc.) and divide by* 3. Why?

Multiply 7824 by $33\frac{1}{3}$. Also 7.824 by $33\frac{1}{3}$.

3) 7 8 2 4 0 0	3) 7 8 2.4
2 6 0 8 0 0, Ans.	2 6 0.8, Ans.

396. To multiply by $66\frac{2}{3}$,

RULE. — *Annex* 00 *(or move, etc.), divide by* 3, *and multiply the quotient by* 2. Why?

Multiply 34278 by $66\frac{2}{3}$. Also 45.3678 by $66\frac{2}{3}$.

3) 3427800	3) 4536.78
1142600	1512.26
2	2
2285200, Ans.	3024.52, Ans.

397. To multiply by $133\frac{1}{3}$,

RULE. — *Annex* 00 *(or move, etc.), divide by* 3, *and add the quotient to the dividend.* Why?

Multiply 78424 by $133\frac{1}{3}$. Also 2.756 by $133\frac{1}{3}$.

3) 7842400	3) 275.6
2614133.333+	91.866+
10456533.333+, Ans.	367.466+, Ans.

398. To multiply by 9, 99, 999, or any number of 9's,

RULE. — *Annex as many ciphers (or move, etc.) as there are* 9's *in the multiplier, and from the result subtract the multiplicand.*

Why? Because annexing 0 multiplies by 10, and, if the multiplicand be subtracted, 9 times the multiplicand will remain; annexing 00 gives 100 times the multiplicand, and taking away once the multiplicand will leave 99 times the multiplicand; etc.

Multiply 7843 by 999. Also 5.69234 by 9999.

7843000	56923.4
7843	5.69234
7835157, Ans.	56917.70766, Ans.

399. How shall we multiply by $12\frac{1}{2}$, $16\frac{2}{3}$, 150, 101, 1001, 19, 91, etc.?

Ex. 1. Multiply 7848 by $12\frac{1}{2}$.

2. Multiply 98.4328 by $12\frac{1}{2}$.

3. Multiply 594327 by 150.
4. Multiply 98.643 by 150.
5. Multiply 378942 by 10001.
6. Multiply 35.6927 by 101.
7. Multiply 46923 by 17.
8. Multiply 46923 by 71.

400. To multiply any whole number $+\frac{1}{2}$ by itself,

RULE.—*Multiply the whole number by the next larger whole number and to the product add* $\frac{1}{4}$.

Ex. 1. Multiply $5\frac{1}{2}$ by $5\frac{1}{2}$. $6 \times 5 + \frac{1}{4} = 30\frac{1}{4}$, Ans.

The reason is found in the following process:—

$$\begin{array}{l} 5 + \frac{1}{2} \\ 5 + \frac{1}{2} \\ \hline 5 \times 5 + \frac{1}{2} \times 5 \\ \qquad\qquad \frac{1}{2} \times 5 + \frac{1}{4} \\ \hline 5 \times 5 + 1 \times 5 + \frac{1}{4}; \end{array}$$

or, $(5 + 1) \times 5 + \frac{1}{4}$,
i. e. $6 \times 5 + \frac{1}{4} = 30\frac{1}{4}$.

First, merely *indicate* the multiplication of $5 + \frac{1}{2}$ by 5 and then by $\frac{1}{2}$, setting the terms of the product as in the margin; then, adding, we find that the product is *five* times 5, *plus once* 5, *plus* $\frac{1}{4}$, which is 6 times $5 + \frac{1}{4}$, a result in accordance with the rule.

Ex. 2. Multiply $9\frac{1}{2}$ by itself.

3. What is the value of a cheese weighing $12\frac{1}{2}$lb. at $12\frac{1}{2}$cts. per pound? Ans. \$$1.56\frac{1}{4}$.

(a) This principle applies equally well to figures of a higher order than units; thus,

Ex. 1. Multiply 75 by 75; i. e. $7\frac{1}{2}$ *tens* by $7\frac{1}{2}$ *tens*.
$80 \times 70 + 5^2 = 5625$, Ans.

2. Multiply 95 by 95. Ans. 9025.
3. Multiply 350 by 350. Ans. 122500.
4. Multiply 7500 by 7500.

401. To multiply any whole number $+ \frac{1}{2}$ by the next larger whole number $+ \frac{1}{2}$,

RULE.—*Multiply the larger whole number by itself and from the product subtract* $\frac{1}{4}$.

Ex. 1. Multiply $6\frac{1}{2}$ by $7\frac{1}{2}$. $7 \times 7 - \frac{1}{4} = 48\frac{3}{4}$, Ans.

2. Multiply $9\frac{1}{2}$ by $10\frac{1}{2}$. Ans. $99\frac{3}{4}$.

(a) The rule applies equally well to examples where the fractions in multiplier and multiplicand vary but $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, etc., from the larger integer, except that the *square of the smaller fraction*, instead of $\frac{1}{4}$, is to be subtracted.

(b) It also applies to figures of a higher order. What is the principle of operation?

Ex. 1. Multiply $9\frac{4}{5}$ by $10\frac{1}{5}$.

$10 \times 10 - (\frac{1}{5})^2 = 99\frac{24}{25}$, Ans.

2. Multiply $3\frac{2}{3}$ by $4\frac{1}{3}$.

3. Multiply 35 by 45. $40 \times 40 - 5^2 = 1575$, Ans.

4. Multiply 27 by 33. $30 \times 30 - 3^2 = 891$, Ans.

5. Multiply 58 by 62.

402. To multiply two decimals together when the product is required to be true only to a certain number of decimal places,

Ex. Multiply 414.793 by 23.7286 and make the product true to 3 decimal places.

UNCONTRACTED PROCESS.

```
         414.793
         23.7286
  --------------
  829586
  1244379
    2903551
      829586
      3318344
        2488758
  --------------
  9842.4571798, Ans.
```

CONTRACTED PROCESS.

```
     414.793
     237286
  ----------
  829586
  1244379
   290355
     8296
     3318
      249
  ----------
  9842.457, Ans.
```

In the *contracted process* it seems most convenient to multiply by the left-hand figure of the multiplier first (392) and we at once perceive that the *whole* multiplicand is to be multiplied by the *integers* of the multiplier, for there are no more decimal places in the multiplicand than are required in the product.

But when we multiply by the .7, we may omit the .003, for .003 × .7 would give a decimal of the *fourth place;* however, as there would be 2 to carry from the product of 3 × 7 to the product of 9 × 7 in the uncontracted process, so 2 must be carried in the contracted process; thus, 7 times 9 are 63, and 2 added will give 65, ∴ 5 is written as the right-hand figure in the 3d partial product. So, when we multiply by .02, both 9 and 3 of the multiplicand may be disregarded, except so far as to determine what to carry to the product of .7 × .02; and, as twice 9 is nearer 20 than 10, there are 2 to carry, which gives 6 for the right-hand figure of the 4th partial product, etc., etc.

Contractions in Division.

403. In *division of decimals*, when the quotient is to be true only to a given number of decimal places, the process may be contracted by dropping, successively, the right-hand figures of the divisor, instead of annexing a cipher or bringing down a figure to the successive partial dividends, taking care to change the right-hand figure of the several products, etc., as would be required if the neglected figures were regarded.

Ex. Divide 7.9362 by 2.7451, true to 4 decimal places.

UNCONTRACTED PROCESS.	CONTRACTED PROCESS.
2.7451) 7.9362 (2.8910	2.7451) 7.9362 (2.8910
54902	54902
24460 0	24460
21960 8	21960
2499 20	2499
2470 59	2470
28 610	28
27 451	27
1 1590	1

For the 1st quotient figure the whole divisor is required; for the 2d quotient figure the 1 of the divisor is dropped, but since the 2d subtrahend is incomplete, its right-hand figure is increased by 1 before subtracting. The same is true of the 3d subtrahend, etc.

404. To Divide by 5,

RULE.—*Multiply the dividend by 2, and point off the right-hand figure.*

Why? Because $\frac{1}{5}$ is twice $\frac{1}{10}$ of a number.

Ex. Divide 7849 by 5.

$$\begin{array}{r} 7849 \\ .2 \\ \hline 1569.8, \text{ Ans.} \end{array}$$

405. To Divide by 25,

RULE.—*Multiply the dividend by* .04. Why?

Ex. Divide 78468 by 25.

$$\begin{array}{r} 78468 \\ .04 \\ \hline 3138.72, \text{ Ans.} \end{array}$$

406. To Divide by $33\frac{1}{3}$.

RULE.—*Multiply the dividend by* .03. Why?

Ex. Divide 3742 by $33\frac{1}{3}$.

$$\begin{array}{r} 3742 \\ .03 \\ \hline 112.26, \text{ Ans.} \end{array}$$

407. To Divide by 125.

RULE.—*Multiply the dividend by* .008. Why?

Ex. Divide 769423 by 125.

$$\begin{array}{r} 769423 \\ .008 \\ \hline 6155.384, \text{ Ans.} \end{array}$$

408. How may we divide by $12\frac{1}{2}$, $16\frac{2}{3}$, $66\frac{2}{3}$, etc.?

Ex. 1. Divide 3550 by $12\frac{1}{2}$.

$$\begin{array}{r} 3550 \\ .08 \\ \hline 284.00, \text{ Ans.} \end{array}$$

2. Divide 869478 by $16\frac{2}{3}$.

IDENTITY OF DIVISION, FRACTIONS AND RATIOS.

409. The dividend in an example in division, the numerator of a fraction and the antecedent of a ratio are identical in office; so, also, are the divisor, denominator and consequent. Hence, whatever operations are performed on dividend and divisor, numerator and denominator, or antecedent and consequent, will affect the quotient, value of the fraction or ratio precisely alike; thus, multiplying the dividend (59, a), numerator (125) or antecedent (244, a), multiplies the quotient, fraction or ratio.

(a) Again multiplying dividend and divisor (60, Cor.), numerator and denominator (133, a, Note 1) or antecedent and consequent (244, e) by the same number, does not alter the quotient, value of the fraction or ratio; ∴ two examples in division may, without altering the quotients, be so changed as to have a *common divisor* or a *common dividend;* two fractions may be reduced to a *common denominator* or a *common numerator;* and two ratios, to a *common consequent* or a *common antecedent.*

COROLLARY TO (a).—Since we may multiply dividend and divisor, numerator and denominator, or antecedent and consequent by *any* number, integral or fractional, it follows that we may *add to* or *subtract from* these corresponding terms *any numbers that have the same ratios*, and the quotient, value of fraction or ratio will remain unchanged; thus,

$$\left.\begin{array}{rr} \text{In Division,} & 15 \div 5 = 3 \\ \text{and} & 6 \div 2 = 3 \\ \therefore\text{, Adding,} & 21 \div 7 = 3 \\ \text{Subtracting,} & 9 \div 3 = 3 \end{array}\right\} \text{ and } \left\{\begin{array}{r} 12 \div 24 = \frac{1}{2} \\ 3 \div 6 = \frac{1}{2} \\ 15 \div 30 = \frac{1}{2} \\ 9 \div 18 = \frac{1}{2} \end{array}\right.$$

$$\left.\begin{array}{rl} \text{In Fractions,} & \frac{5}{15} = \frac{1}{3} \\ \text{and} & \frac{2}{6} = \frac{1}{3} \\ \therefore\text{, Adding,} & \\ \text{and Subtracting,} & \dfrac{5+2}{15+6} = \dfrac{5-2}{15-6} = \frac{1}{3} \end{array}\right\} \text{ and } \left\{\begin{array}{l} \frac{9}{3} = 3 \\ \frac{6}{2} = 3 \\ \dfrac{9+6}{3+2} = \dfrac{9-6}{3-2} = 3 \end{array}\right.$$

$$\left.\begin{array}{rr} \text{In Ratio,} & 15 : 5 = 3 \\ \text{and} & 3 : 1 = 3 \\ \therefore\text{, Adding,} & 18 : 6 = 3 \\ \text{Subtracting,} & 12 : 4 = 3 \end{array}\right\} \text{ and } \left\{\begin{array}{r} 3 : 12 = \frac{1}{4} \\ 2 : 8 = \frac{1}{4} \\ 5 : 20 = \frac{1}{4} \\ 1 : 4 = \frac{1}{4} \end{array}\right.$$

REMARK.—It is on this principle that the rule for reducing pence and farthings to the decimal of a pound *by inspection*, is founded; thus, 1 qr. $= \frac{1}{960}$£, but if 960 be increased by $\frac{1}{24}$ of itself the sum will be 1000; ∴, increasing the numerator and denominator, each, by $\frac{1}{24}$ of itself, we shall have $\frac{1}{960} = \frac{1\frac{1}{24}}{1000}$, i. e., 1 qr. $= \frac{1\frac{1}{24}}{1000}$£, and, for a like reason, 2 qr. $= \frac{2\frac{2}{24}}{1000}$£ $= \frac{2\frac{1}{12}}{1000}$£, 24 qr. $= \frac{24\frac{24}{24}}{1000}$£ $= \frac{25}{1000}$£, 48 qr. $= \frac{50}{1000}$£, etc.

Now 2s. $= \frac{2}{20}$£ $= \frac{1}{10}$£ $=$ .1£, 4s. $=$ .2£, 16s. $=$ .8£, etc.; and 1s. $= \frac{1}{20}$£ $=$ .05£. Hence,

410. To reduce shillings, pence and farthings to the decimal of a pound, *by inspection*,

RULE.—*Write half the greatest even number of shillings as so many tenths of a pound; write the odd shilling, if there be one, as* .05 *of a pound; and write the number of farthings in the given pence and farthings, increased by* 1 *if the number is* 12 *or more, and by* 2 *if it is* 36 *or more, as so many thousandths of a pound. The sum of these will not vary more than* $\frac{1}{2}$ *of* .001£ *(or less than* $\frac{1}{2}$ *qr.) from the true value of the given shillings, pence and farthings.*

Ex. 1. Reduce 15s. 3d. 2qr. to the decimal of a pound by inspection?

$$\begin{array}{rl} 14\text{s.} & = .7\ \ \text{£} \\ 1\text{s.} & = .05\ \text{£} \\ 3\text{d. } 2\text{qr.} & = .015\text{£, nearly,} \\ \hline \therefore\ 15\text{s. } 3\text{d. } 2\text{qr.} & = .765\text{£, nearly, Ans.;} \end{array}$$

Or, more briefly and compactly,

$$\begin{array}{rl} 15\text{s.} & = .75\ \text{£} \\ 3\text{d. } 2\text{qr.} & = .015\text{£, nearly,} \\ \hline \therefore\ 15\text{s. } 3\text{d. } 2\text{qr.} & = .765\text{£, nearly, Ans. as before.} \end{array}$$

Ex. 2. Reduce 19s. 8d. 1qr. to the decimal of a pound.

Ans. .984£.

3. Reduce 12s. 11d. 3qr. to the decimal of a pound.

411. Reversing this operation we may reduce the first 3 figures of the decimal of a pound, back to shillings, pence and farthings; thus,

Ex. 1. Reduce .875£ to shillings and pence.

$$\begin{array}{rl} .8\ \ \ \pounds = & 16s. \\ .05\ \pounds = & 1s. \\ .025\pounds = & 6d. \\ \hline \therefore\ .875\pounds = & 17s.\ 6d. \end{array}$$ Hence,

To reduce t first 3 figures of a decimal of a pound to shillings, pence an arthings, by inspection,

Rule.—*Do e the tenths for shillings; if the hundredths b 5 or more, add nother shilling; then, after the* .05 *is deducted the remaining figures of the* 2*d and* 3*d places, abating* 1 *when the remainder is* 12 *or more, and* 2 *when* 36 *or more, will represent the farthings, which may be reduced to pence and farthings.*

2. Reduce .784£ to shillings, pence and farthings.

Ans. 15s. 8d. 1qr.

3. Reduce .247£ to shillings, pence and farthings.

Barter.

412. *Barter* is an exchange of commodities in trade.

Questions in barter are solved by analysis.

Ex. 1. How much coffee, at 25c. per pound, must be given in exchange for 300 pounds of sugar, at 15c. per pound?

Ans. 180lb.

2. How many bushels of oats, at 50c. per bushel, are equal in value to 1000 bushels of wheat at $2.37\frac{1}{2}$ per bushel?

Ans. 4750

3. A has flour worth \$10 per barrel; but, in exchanging it with B, for broadcloth, he asks \$12. Now, B's broadcloth being worth \$4 per yard, what shall he charge for it that he may not suffer loss? Ans. \$4.80 per yard.

4. C has $193\frac{7}{8}$ pounds of tea, worth $62\frac{1}{2}$c. per pound, which he will put at $56\frac{1}{4}$c. provided he can get coffee, worth 25c. per pound, for 23c. Does he gain or lose, and what per cent.? Ans. Loses 2 per cent.

Practice.

413. *Practice* is a mode of finding the value of any number of articles at any price, by assuming the value of the whole or a part, at the given or some other price, and then modifying the assumption according to circumstances.

Ex. 1. What is the value of 9a. 3r. 20rd. of land, at \$40 per acre?

$$\begin{array}{rl} \$40 & \\ 9 & \\ \hline \$360 = & \text{value of 9a.} \\ 20 = & \text{“ “ 2r.} = \tfrac{1}{2}\text{a.} \\ 10 = & \text{“ “ 1r.} = \tfrac{1}{2} \text{ of 2r.} \\ 5 = & \text{“ “ 20rd.} = \tfrac{1}{2}\text{r.} \\ \hline \$395 = & \text{“ “ 9a. 3r. 20rd., Ans.} \end{array}$$

2. What is the value of 356 barrels of flour, at $\$9\frac{1}{2}$ per barrel?

$$\begin{array}{rl} \$3560 = & \text{value at \$10} \\ 178 = & \text{“ “ \$}\tfrac{1}{2} \\ \hline \$3382 = & \text{“ “ \$}9\tfrac{1}{2}\text{, Ans.} \end{array}$$

Series.

414. A *series* consists of 3 or more terms, following each other in accordance with some law (109*).

415. An *infinite series* is one which is without end.

416. The sum of an ascending infinite series is infinite; i. e. great beyond limits.

417. The sum of a descending infinite series in geometrical progression, may be found by the rule in Art. 363, except that in the infinite series the least term may be considered 0, and ∴ disregarded.

A quantity that is small beyond any determinate limits is an *infinitesimal;* as, e. g. the smaller terms of a descending infinite series.

Ex. 1. What is the sum of the infinite series, 6, 2, $\frac{2}{3}$, $\frac{2}{9}$, etc.?

$$6 \div 2 = 3 ; \text{ and } 3 + 6 = 9, \text{ Ans.}$$

2. What is the sum of the infinite series, 1, $\frac{1}{2}$, $\frac{1}{4}$, etc.?

Ans. 2.

3. What is the sum of the infinite series, $\frac{3}{10}$, $\frac{3}{100}$, $\frac{3}{1000}$, etc.?

Ans. $\frac{1}{3}$.

418. There are various methods of finding the sums of different series, but they are Algebraic and cannot be investigated in this treatise. Rules for summing two species of series, only, will be given here.

(a) To find the sum of the squares of any number of terms in the natural series, 1, 2, 3, 4, etc.,

RULE.—*Multiply the number of terms in the series by that number plus one; then multiply the product by twice the number plus one, and* $\frac{1}{6}$ *of the product will be the sum sought.*

Ex. 1. What is the sum of 12 terms of the series, 1^2, 2^2, 3^2, etc.?

$$\frac{12 \times (12 + 1) \times (24 + 1)}{6} = 650, \text{ Ans.}$$

2. What is the sum of 12 terms of the series, 6^2, 7^2, 8^2, etc.?

Ans. 1730.

The rule is not directly applicable to this example, but we must get the sum of 17 terms of the series, 1^2, 2^2, 3^2, etc., and also of 5 terms, and the difference of these sums will be the

sum sought. A practical application of this rule is in finding the number of cannon balls in a square pile, e. g. 12 balls on a side in the lower layer, then 11 in the 2d layer, 10 in the 3d, etc.

(b) To find the sum of the cubes of any number of terms in the natural series, 1, 2, 3, etc.,

RULE 1.—*Multiply the number of terms plus one, by half the number of terms, and square the product;* or,

RULE 2.—*Square the sum of the terms.*

Ex. 1. What is the sum of 8 terms of the series, 1^3, 2^3, 3^3, etc.?

$$\overline{(8+1)\times 4}^{\,2}=1296, \text{ Ans.}$$

2. What is the sum of 8 terms of the series, 5^3, 6^3, 7^3, etc.?

Ans. 5984.

Are the rules directly applicable to this example?

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§ 47. CIRCULATING DECIMALS.

419. A CIRCULATING DECIMAL (157, c) is a decimal in which a certain figure or a succession of figures is repeated over and over again, *perpetually.*

420. Such a *circulate* or *repetend* is obtained by reducing a vulgar fraction to a decimal (158) whenever there is any prime factor except 2 and 5 in the denominator and not in the numerator; e. g. $\frac{1}{9}=.1111$ etc.; the 1 to be repeated perpetually. $\frac{7}{9}=.7777$ etc. $\frac{8}{11}=.727272$ etc. $\frac{41}{333}=.123123$, etc.

421. When only one figure is repeated, it is called a *simple repetend;* thus, $\frac{2}{9}=.222$ etc., a simple repetend.

422. When two or more figures are repeated, it is a *compound repetend;* thus, $\frac{8}{11}=.7272$ etc., a compound repetend.

423. Instead of *repeating* a figure, it is written once and a point is placed over it, thus, $\frac{5}{9}=.\dot{5}$. If the repetend is com

pound, points are placed over the first and last figure of the circulate; thus, $\frac{41}{333} = .\dot{1}2\dot{3}$.

424. When other decimal figures occur before the repeating figures commence, it is called a *mixed repetend;* thus, $\frac{5}{12} = .41666$ etc. $= .41\dot{6}$, a mixed repetend. The figures preceding the circulate are the *finite* part of the expression.

425. A *perfect repetend* is one in which the number of figures in the circulate is one less than the number of units in the denominator of the equivalent vulgar fraction; thus, $\frac{1}{7}$ gives the perfect repetend $.\dot{1}4285\dot{7}$; $\frac{1}{29}$ gives $.\dot{0}34482758620689655172413793\dot{1}$.

426. There can be no more figures in a repetend than one less than the number of units in the denominator of the equivalent vulgar fraction, reduced to its lowest terms; for, in dividing the numerator by the denominator, each remainder must be less than the denominator, and $\therefore$ there can never be more *different* remainders than there are units in the denominator, less one; and whenever a remainder like a preceding remainder occurs, the quotient figures must begin to repeat.

427. Since $\frac{1}{9} = .\dot{1}$ it follows (by multiplying each member of the equation by 2, 3, 7, etc., and transposing the members) that $.\dot{2} = \frac{2}{9}$, $.\dot{3} = \frac{3}{9}$, $.\dot{7} = \frac{7}{9}$; i. e. a simple repetend is reduced to an equivalent vulgar fraction by writing the repetend for a numerator and 9 for a denominator.

428. Since $\frac{1}{99} = .\dot{0}\dot{1}$, it follows that $.\dot{0}\dot{2} = \frac{2}{99}$, $.\dot{2}\dot{3} = \frac{23}{99}$, etc. Similar reasoning will show us that any compound repetend may be reduced to an equivalent vulgar fraction by writing the repetend for the numerator and as many 9's as there are figures in the repetend for the denominator.

429. In a *mixed* repetend the figures preceding the circu

late have just the same value they would have if no circulate followed them; thus, in $.2\dot{7}$, the $.2$ is $\frac{2}{10}$; in $.15\dot{3}$, the $.15$ is $\frac{15}{100}$, etc. Moreover, the value of the circulate is affected by having other decimal figures precede it, just as the value of any decimal figure is affected by having other decimal figures precede it; thus, in $.2\dot{7}$, the $\dot{7}$ is only $\frac{1}{10}$ as great as in the expression $2.\dot{7}$; in $.15\dot{3}$, the $\dot{3}$ is only $\frac{1}{100}$ as great as in $15.\dot{3}$; but in $2.\dot{7}$ the $\dot{7}$ is equal to $\frac{7}{9}$; $\therefore$, in $.2\dot{7}$, the $\dot{7}$ is equal to $\frac{7}{9} \times \frac{1}{10} = \frac{7}{90}$; for a like reason, in $.15\dot{3}$, the $\dot{3}$ is equal to $\frac{3}{900}$, etc. Hence,

430. To find the value of a mixed circulate,

RULE.—*Find the value of the finite part and of the repetend separately, and add the two together.*

Ex. 1. What is the value of $.27\dot{5}$?

$$.27\dot{5} = .27 + \tfrac{5}{900} = \tfrac{243}{900} + \tfrac{5}{900} = \tfrac{248}{900} = \tfrac{62}{225}, \text{ Ans.}$$

2. What is the value of $.42\dot{1}2\dot{3}$? Ans. $\frac{14027}{33300}$

431. *Perfect repetends* have some very curious properties which have been very happily presented by G. R. Perkins, Esq., an eminent American Teacher and Author of an able series of mathematical works.

(a) If the last half of the figures of a perfect repetend be written in order under the first half and added to the figures in the first half, the sum will be a succession of 9's; thus, the fraction $\frac{1}{23} = .\dot{0}43478260869565217391\dot{3}$, and this repetend, written and added as suggested, will give

$$\begin{array}{r} 0\,4\,3\,4\,7\,8\,2\,6\,0\,8\,6 \\ 9\,5\,6\,5\,2\,1\,7\,3\,9\,1\,3 \\ \hline 9\,9\,9\,9\,9\,9\,9\,9\,9\,9\,9 \end{array}$$

(b) If the remainders obtained in reducing the vulgar fraction to a repetend be written in the same way and added, each

sum will be the denominator of the vulgar fraction; thus, the remainders in reducing $\frac{1}{23}$ are

	10,	8,	11,	18,	19,	6,	14,	2,	20,	16,	22,
	13,	15,	12,	5,	4,	17,	9,	21,	3,	7,	1, which, added,
give	23,	23,	23,	23,	23,	23,	23,	23,	23,	23,	23.

(c) If we subtract the unit figure of the denominator of the vulgar fraction from 10 and multiply any figure of the repetend by the remainder, the unit figure of the product will be the unit figure of the corresponding remainder; thus,

in $\frac{1}{23} = .0$; 4, 3, 4, 7, 8, 2, etc., figures of repetend.
$10 - 3 = 7$

0, 8, 1, 8, 9, 6, 4, etc., unit figures of products and remainders.

(d) The repetend $.\dot{0}43478260869565217391\dot{3}$ is not only the circulate equivalent to $\frac{1}{23}$, but also, beginning at different points, the same figures in the same order of succession, will be the repetend equal to $\frac{2}{23}$, $\frac{3}{23}$, $\frac{4}{23}$, etc., up to $\frac{22}{23}$; thus, $\frac{10}{23} = .\dot{4}3478$, etc., which begins with the 2d figure of the circulate equal to $\frac{1}{23}$. Again, $\frac{8}{23} = .\dot{3}4782$, etc., which begins with the 3d figure of the circulate equal to $\frac{1}{23}$; etc.

It will be observed that the numerator of the fraction equal to the several repetends beginning with the successive figures of $.\dot{0}43478$, etc., is the remainder left when the preceding figure of the circulate was obtained; thus, when the first 4 of the circulate was obtained, 8 was the remainder, and 8 is the numerator of the fraction equal to the circulate $.\dot{3}4782$, etc.

432. The following are all the fractions whose numerators are a unit and denominators less than 100, which give *perfect* repetends, viz. $\frac{1}{7}$, $\frac{1}{17}$, $\frac{1}{19}$, $\frac{1}{23}$, $\frac{1}{29}$, $\frac{1}{47}$, $\frac{1}{59}$, $\frac{1}{61}$ and $\frac{1}{97}$.

433. When many figures in the decimal are required, the process may be shortened as follows: —

Ex. Reduce $\frac{1}{29}$ to a decimal fraction.

```
29 ) 1.00 ( .03448 8/29
      87
     ---
      130
      116
      ---
       140
       116
       ---
        240
        232
        ---
          8
```

Having continued the operation by the usual mode until a small remainder is obtained, we write that remainder over the divisor, and annex the vulgar fraction so formed to the decimal, and so obtain

(1) $\frac{1}{29}=.03448\frac{8}{29}$. Multiplying this by 8, we have

(2) $\frac{8}{29}=.27586\frac{6}{29}$. Substituting this value of $\frac{8}{29}$ in (1), we have

(3) $\frac{1}{29}=.0344827586\frac{6}{29}$. Multiplying this by 6, gives

(4) $\frac{6}{29}=.2068965517\frac{7}{29}$. Substituting in (3), gives

(5) $\frac{1}{29}=.03448275862068965517\frac{7}{29}$. Multiplying by 7, we have

(6) $\frac{7}{29}=.24137931034482758620\frac{20}{29}$. Substituting in (5), we have

(7) $\frac{1}{29}=.0344827586206896551724137931,034482758620\frac{20}{29}$.

There will be nothing gained by continuing this process; for, by Art. 426, there cannot be more than 28 figures in the circulate. Indeed, it will be seen that the figures begin to repeat at the 29th place.

§ 48. CONTINUED FRACTIONS.

434. A CONTINUED FRACTION is a fraction whose numerator is a unit and whose denominator is a whole number plus a fraction, and this latter fraction has a unit for its numerator and its denominator is a whole number plus a fraction, etc., etc.; thus,

$$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{3+}}}}}\text{, etc., is a continued fraction.}$$

435. A common fraction may be reduced to a continued fraction as follows :—

$$\frac{17}{43}=\cfrac{1}{2+\frac{9}{17}} \text{ and } \frac{9}{17}=\cfrac{1}{1+\frac{8}{9}}; \therefore \frac{17}{43}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{8}{9}}}$$

$$\text{Again, } \frac{8}{9}=\cfrac{1}{1+\cfrac{1}{8}}; \therefore \frac{17}{43}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{8}}}}.$$

This process consists in, first, dividing both terms of $\frac{17}{43}$ by 17, the numerator (129), which gives $\frac{17}{43}=\cfrac{1}{2+\cfrac{9}{17}}$; then, in like manner reducing $\frac{9}{17}$ to $\cfrac{1}{1+\cfrac{8}{9}}$, and substituting this value of $\frac{9}{17}$ in the expression $\cfrac{1}{2+\cfrac{9}{17}}$, giving $\frac{17}{43}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{8}{9}}}$, etc., etc.

436. In any continued fraction, $\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{9}}}}$, the several simple fractions are called *integral fractions* because their denominators are integers; thus, $\frac{1}{2}$ in the above is the 1st integral fraction; $\frac{1}{1}$ is the 2d; $\frac{1}{1}$ is the 3d; and $\frac{1}{9}$ is the 4th integral fraction. $\frac{1}{2}$ is also called the 1st *approximating* or *converging fraction;* $\cfrac{1}{2+\cfrac{1}{1}}$ is the 2d approximating fraction, etc., etc.

437. Continued fractions have many remarkable properties and important relations to mathematical science. The investigation of these properties and relations being Algebraic is omitted, and a few of them only are here enumerated.

(a) The *numerator* of the 2d converging fraction, when reduced to a simple form, is the numerator of the 1st multiplied by the denominator of the 2d integral fraction. The *denominator* of the 2d is equal to the denominator of the 1st multiplied by the 2d integral denominator, plus the 2d integral numerator. Again, the numerator of the 3d converging fraction is equal to the product of the numerator of the 2d by the 3d *integral* denominator, plus the 1st numerator. The denominator of the 3d is equal to the denominator of the 2d multiplied by the 3d integral denominator, plus the 1st denominator.

And, *generally*, of any 3 successive converging fractions, the numerator of the 3d is equal to the numerator of the 2d multiplied by the last integral denominator considered, plus the numerator of the 1st of the 3 converging fractions. The denominator of the 3d is equal to the denominator of the 2d multiplied by the 3d *integral* denominator, plus the 1st of the 3 converging denominators.

Ex. What are the successive converging fractions in the continued fraction

$$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{4}}}}?$$

$$\frac{1}{2}=\text{1st.} \qquad \cfrac{1}{2+\cfrac{1}{3}}=\frac{3}{7}=\text{2d}$$

$$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{5}}}=\frac{16}{37}=\text{3d.}$$

$$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{4}}}}=\frac{67}{155}=\text{4th.}$$

(b) The successive converging fractions are alternately too large and too small; the 1st, 3d, 5th, etc., too large; the 2d, 4th, 6th, etc., too small. The reason is obvious. The 1st denominator, 2, is not large enough, ∴ the *fraction*, $\frac{1}{2}$, is too large (59, f). The 2d denominator, 3, is not large enough; ∴ $\frac{1}{3}$ is too large, and this added to the denominator, 2, makes that denominator too large and ∴ the fraction too small, etc., etc.

(c) Any one of the converging fractions differs from the fraction from which it is derived by less than the square of the reciprocal of its denominator; thus, $\frac{3}{7}$ differs from the value of the above continued fraction by less than $\left(\frac{1}{7}\right)^2=\frac{1}{49}$. The successive approximating fractions approach nearer and nearer to the true value.

(d) The numerators of any two successive converging frac-

tions, when reduced to a common denominator, differ from each other by a unit; thus, $\frac{16}{37}-\frac{3}{7}=\frac{112-111}{259}=\frac{1}{259}$.

(e) The terms of every approximating fraction are mutually prime (94, a, Note 2).

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§ 49. CHANGES IN PROPORTION.

438. The *antecedent* and *consequent* of either couplet are called *analogous terms;* and the two *antecedents or* the two *consequents, homologous terms.*

439. Any change in the *order* or *magnitude* of the terms of a proportion which does not affect the *equality of the ratios,* does not destroy the proportion. Hence,

(a) (1) Given	8 : 4 : : 6 : 3
(2) Alternating (1)	8 : 6 : : 4 : 3
(3) Inverting (1)	4 : 8 : : 3 : 6
(4) Alternating (3)	4 : 3 : : 8 : 6
(5) Inverting (1) and transposing couplets	3 : 6 : : 4 : 8
(6) Alternating (5)	3 : 4 : : 6 : 8
(7) Inverting (5)	6 : 3 : : 8 : 4
(8) Alternating (7)	6 : 8 : : 3 : 4

These 8 forms are all manifestly true proportions, for, in making the changes, we have steadily kept in view the simplest test of proportionality (254).

Can these 4 numbers be written in any other order? How many (379)? Are the numbers in any of these other orders in proportion?

(b) If any two analogous terms, or any two homologous terms

be multiplied or divided by the same number, the proportion will be preserved; thus,

$$\begin{array}{rcccccccc} & 24 & : & 12 & :: & 16 & : & 8 \\ \text{Multiplying by} & 2 & & 2 & & & & \\ \hline & 48 & : & 24 & :: & 16 & : & 8 \end{array} \qquad \begin{array}{ccccccc} 24 & : & 12 & :: & 16 & : & 8 \\ & & & & 3 & & 3 \\ \hline 24 & : & 12 & :: & 48 & : & 24 \end{array}$$

$$\begin{array}{rcccccccc} \text{Again,} & 24 & : & 12 & :: & 16 & : & 8 \\ \text{Dividing by,} & 3 & & 3 & & & & \\ \hline & 8 & : & 4 & :: & 16 & : & 8 \end{array} \qquad \begin{array}{ccccccc} 24 & : & 12 & :: & 16 & : & 8 \\ & & & & 4 & & 4 \\ \hline 24 & : & 12 & :: & 4 & : & 2 \end{array}$$

$$\begin{array}{rcccccccc} \text{Again,} & 24 & : & 12 & :: & 16 & : & 8 \\ & 3 & & 3 & & 2 & & 2 \\ \hline & 72 & : & 36 & :: & 8 & : & 4 \end{array} \qquad \begin{array}{ccccccc} 24 & : & 12 & :: & 16 & : & 8 \\ 6 & & 6 & & 3 & & 3 \\ \hline 4 & : & 2 & :: & 48 & : & 24 \end{array}$$

None of these changes affect either *ratio* (244, e), ∴ the proportion is preserved.

$$\begin{array}{rcccccccc} \text{Again,} & 24 & : & 12 & :: & 16 & : & 8 \\ & 2 & & & & 2 & & \\ \hline & 48 & : & 12 & :: & 32 & : & 8 \end{array} \qquad \begin{array}{ccccccc} 24 & : & 12 & :: & 16 & : & 8 \\ & & 3 & & & & 3 \\ \hline 24 & : & 36 & :: & 16 & : & 24 \end{array}$$

Do the above operations affect the *ratios?* How? Why (244, a and c)? Do they destroy the *proportion?* Why (250)? Are there any other ways of multiplying or dividing the terms of a proportion without destroying the proportion? What are they?

(c) If the terms of one proportion are multiplied or divided by the corresponding terms of another, the products or quotients will be proportional; thus,

(1)	$12 : 4 :: 48 : 16$
(2)	$8 : 2 :: 4 : 1$
(3) The terms of (1) × by those of (2),	$96 : 8 :: 192 : 16$
(4) " " " (1) ÷ " " " (2),	$1\frac{1}{2} : 2 :: 12 : 16$

The reason is plain. In (3) the ratios are compounded of equal pairs of ratios, and are ∴ equal (249). (4) is the reverse of (3).

COROLLARY TO (c).—If the terms of a proportion be squared, cubed, etc., or if the 2d, 3d, etc., roots be taken, the powers or roots will be proportional; thus,

If $3 : 2 : : 9 : 6$, then $3^3 : 2^3 : : 9^3 : 6^3$. Why (249, a)?

If $100 : 25 : : 36 : 9$, then $\sqrt{100} : \sqrt{25} : : \sqrt{36} : \sqrt{9}$. Why?

(d) If two proportions have a *common couplet*, the remaining couplets will constitute a proportion; for two ratios that are respectively equal to the same ratio are equal to each other; thus,

If	$6 : 3 : : 18 : 9$	or	$8 : 2 : : 12 : 3$
and	$6 : 3 : : 20 : 10$	and	$12 : 3 : : 16 : 4$
then	$18 : 9 : : 20 : 10$	and	$8 : 2 : : 16 : 4$

Again, if two analogous terms of one proportion are like two homologous terms of another, then the four remaining terms will be proportional; for, by alternation, the like terms may be made analogous; thus,

$$\left.\begin{array}{l}\text{Let } 12 : 4 : : 15 : 5\\ \text{and } 12 : 6 : : 4 : 2\end{array}\right\}\begin{array}{c}\text{then by}\\ \text{alternation}\end{array}\left\{\begin{array}{l}12 : 4 : : 15 : 5\\ 12 : 4 : : 6 : 2\end{array}\right.$$

$\therefore$ by the above, $15 : 5 : : 6 : 2$.

The comparison of proportions having like terms may be varied in many ways.

(e) The analogous or homologous terms of a proportion may be increased or diminished by terms having the same ratio (409), without destroying the proportion; thus, if $20 : 5$ and $12 : 3$ have the same ratio as $4 : 1$, then $20 + 12 : 5 + 3 : : 4 : 1$ and $20 - 12 : 5 - 3 : : 4 : 1$, etc., etc.

440. The changes that may be made on the terms of a proportion are *very numerous*, but they are reducible to a few general principles; as,

(a) *By changing the order of the terms,*

(b) *By multiplying or dividing by the same number,*

(c) *By multiplying or dividing the terms of one proportion by those of another,*

(c, Cor.) *By involving or evolving the terms,*

(d) *By comparing proportions which have like terms,*

(e) *By adding or subtracting terms of equal ratios.*

REMARK.—A familiar acquaintance with these changes will greatly facilitate the study of Algebra and the Higher Mathematics.

441. *A continued proportion* is one composed of several ratios, in which the consequent of the 1st ratio is the antecedent of the 2d; the consequent of the 2d, the antecedent of the 3d, etc.; thus,

$$2 : 4 : : 4 : 8 : : 8 : 16 : : 16 : 32, \text{ etc.}$$

442. In continued proportion the number of *different* quantities is one greater than the number of couplets and the 1st is to the 3d, as the square of any one of the antecedents is to the square of its consequent; the 1st is to the 4th as the cube of either antecedent to the cube of its consequent, etc. etc.; thus, in the continued proportion given above $2 : 8 : : 2^2 : 4^2$ or as $4^2 : 8^2$, etc.

Again, $2 : 16 : : 2^3 : 4^3$ or as $16^3 : 32^3$, etc.

443. Three or four quantities are in *harmonical* or *musical proportion* when the first is to the last as the difference between the first two is to the difference between the last two; thus, 20, 16, 12, and 10 are in harmonical proportion, for $20 : 10 : : 20 - 16 : 12 - 10$.

444. If the reciprocals of any arithmetical series of integral numbers be reduced to a common denominator, any three consecutive numerators will be in harmonical proportion; thus, take the series 2, 5, 8, and 11, whose reciprocals are $\frac{1}{2}$, $\frac{1}{5}$, $\frac{1}{8}$, and $\frac{1}{11}$ $= \frac{440}{880}$, $\frac{176}{880}$, $\frac{110}{880}$, and $\frac{80}{880}$, and the numerators are in harmonical proportion; for, $440 : 110 : : 440 - 176 : 176 - 110$ and $176 : 80 : : 176 - 110 : 110 - 80$.

Again, take 9, 7, 5, 3, 1, whose reciprocals are $\frac{1}{9}$, $\frac{1}{7}$, $\frac{1}{5}$, $\frac{1}{3}$, $\frac{1}{1}$ = $\frac{105}{945}$, $\frac{135}{945}$, $\frac{189}{945}$, $\frac{315}{945}$, $\frac{945}{945}$, and we have $105 : 189 :: 135 - 105 \cdot 189 - 135$, etc. etc.

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§ 50. THE CALENDAR AND CHRONOLOGICAL PROBLEMS.

445. The only natural and obvious divisions of time are days, months (moons), and years. Other distinctions, such e. g. as hours, weeks, centuries, etc., are artificial, and consequently different nations have made different divisions, and dated their reckoning from different epochs, and thus there has been very great confusion in respect to dates.

446. The Solar Year is the time occupied by the earth in making one revolution in its orbit, as, e. g. in passing from the vernal equinox (the time of equal day and night in the spring) round to that point again.

447. The most ancient nations, by noting the time when a vertical rod, called the stylus, cast the shortest shadow at noon in successive years, discovered that the solar year consisted of 365 entire days; but, by the aid of modern science, the year is found to consist of very nearly 365 days, 5 hours, 48 minutes, and 49.62 seconds.

448. Now, if a year were just 365 days, and the stylus cast the shortest shadow on the 21st of June this year, it would do the same each succeeding year, perpetually; but as a year is nearly 365$\frac{1}{4}$ days, the shortest shadow, after the lapse of four years, would not be cast until June 22d, and not until June 23d in 4 years more; and thus the summer solstice, (the time of the

shortest shadow,) would occur on every successive day in the year.

449. To avoid this confusion, the Roman Emperor, Julius Cæsar, in the year 46 before Christ, introduced a day in February every 4th year, and thus made every fourth year (bissextile or leap year) consist of 366 days; but this correction was too great by more than 11 minutes a year, and consequently in about 129 years the summer solstice would occur one day earlier, say June 20th, and in 129 years more it would occur June 19th, and so on.

450. At the time of the Council of Nice, A. D. 325, the vernal equinox was known to be on the 21st of March; but, by following the rule given by Cæsar, making every 4th year consist of 366 days, the Calendar had been deranged 10 days before the time of Pope Gregory XIII, who, in the year 1582, to restore the equinox to the 21st of March, decreed that the year should be brought forward 10 days, by calling the 5th of October the 15th, and the succeeding days in order, 16th, 17th, etc., and, to prevent similar confusion afterwards he made this

RULE.—*Every year whose number is divisible by* 4, *except those divisible by* 100 *and not by* 400, *shall consist of* 366 *days and all others of* 365 *days.*

451. A Julian period of 400 years is thus three days longer than the same period by the Gregorian rule; but this is not quite so much as the actual difference between 400 Julian and 400 solar years; for 400 Julian years are 146100 days, while 400 solar years are 146096.896 + days, and 146100 — 146-096.896 = 3.1 +. Thus even a *Gregorian* period of 4000 years is a little more than 1 day longer than 4000 solar years. Had Gregory extended his rule by making the years 4000, 8000, etc., to consist of 365 instead of 366 days, as they now do by his rule, the error would be less than 1 day in 100000 years.

452. Dates by the Julian Calendar are in *old style* (O S.), and those by the Gregorian are in *new style* (N. S.).

453. England did not adopt the correction made by Gregory until 1752, when the error in the Julian Calendar was 11 days. Then, by act of Parliament, the year was brought forward 11 days, by calling the 3d of September the 14th, and by the same act the year, which had commenced on the 25th of March, was made to commence on the 1st of January, thus making the year 1751 consist of only about 9 months.

454. In consequence of correcting the calendar, English dates in old style and new, differ from each other not only in the day of the month, but, for that part of the year preceding March 25th, they also differ in the number of the year; e. g. Washington was born Feb. 11, 1731, O. S., but, by new style, the date would have been Feb. 22, 1732, and it is usual, in such cases, to write both years; thus, Feb. 11, 1731–2, O. S.; or thus, Feb. $\frac{11}{22}$, 173$\frac{1}{2}$.

455. In constructing a calendar, the first problem is to connect the *week* with the *year;* i. e. to find the day of the week corresponding with any given day of any year. To do this the first 7 letters of the alphabet are used, A to designate the 1st day of Jan., B, C, D, E, F and G for the 2d, 3d, 4th, 5th, 6th and 7th, and then A is repeated for the 8th and so on through the year. Consequently, *one* of these 7 letters must stand for *Sunday.* This letter is called the *Sunday Letter* or *Dominical** *Letter;* thus, if Jan. begins on Sunday, A is the dominical letter for that year; if Jan. begins on Monday, the 1st Sunday will be the 7th day, and ∴ G, the 7th letter, will be the dominical letter; etc., etc. Now, Jan. 1, 1854, was Sunday, and ∴ A was the dominical letter for 1854; and as a common year consists of 52 weeks and 1 day (= 365 days), 1854 also *closed* on Sunday; hence, 1855 began on Monday and G was the dominical letter for 1855. Again, 1855 being a common year, closed on Monday, and 1856 began on Tuesday, and ∴ F was its dominical

* *Dominical,* from the Latin *dominus, lord.*

letter. Thus, if every year consisted of 52 weeks and 1 day the dominical letters for successive years would be the 1st 7 letters of the alphabet in retrograde order, G, F, E, D, C, B, A; but, as 1856 was leap year, it consisted of 52 weeks and 2 days, and ∴ closed on Wednesday instead of Tuesday and consequently 1857 came in on Thursday, and D, instead of E, is the dominical letter for 1857.

Thus there is a break in the order of the dominical letters once in 4 years, and the series cannot return to its first state (i. e. the same days of the months return to the same days of the week) until 4 times 7 or 28 years, and even this order of succession, in the Gregorian calendar, will be broken at the close of the century, because the hundredth year, though divisible by 4, is yet a common year (unless divisible by 400), and the usual break at the end of every 4 years will therefore not occur; but this is easily obviated by a correction at the beginning of the century.

456. It is customary to assign *two* dominical letters to each leap year, one for Jan. and Feb. and the next preceding letter of the alphabet for the rest of the year. This may be explained by designating the 28th and 29th of Feb. by the same letter; thus, in 1856, F is the dominical letter for Jan. and Feb., and ∴ Thursday, the 28th of Feb., is designated by C, and if Friday, the 29th, is *also* represented by C, then Saturday, March 1st, will be represented by D and Sunday by E. The dominical letter for the *greater portion* of leap year is the one determined in the following table.

457. This table was prepared by Samuel Maynard, an English mathematician, and by it we may readily find the dominical letter for any year and also the day of the week for any day of the year, both for old style and new, before and after Christ, supposing these styles extended backward to the beginning of time and onward indefinitely.

TABLE FOR VERIFYING DATES.

Directions for using the Table.—If the given date be old style, and less than 700 years, or new style and less than 400 years, it will be the tabular date for its respective style; but if the date be old style and 700 or more, divide by 700, or if new style and 400 or more, divide by 400, the remainder will be the tabular date, the centuries of this remainder or tabular date being given in the upper part of the middle portion of the table against their title, for new style (N. S.), and old style (O. S.), as well for before Christ (B. C.), as after Christ (A. C.), the remaining years will be found on the right or left according as the given date is before Christ or after, then the Dominical letter of the given date will be found on the same horizontal line with these remaining years, in the same vertical column with their centuries.

Remaining Years above Centuries after Christ.

·0	6		17	23	·28	34		45	51	·56	62		73	79	·84	90	
1	7	·12	18		29	35	·40	46		57	63	·68	74		85	91	·96
2		13	19	·24	30		41	47	·52	58		69	75	·80	86		97
3	·8	14		25	31	·36	42		53	59	·64	70		81	87	·92	98
	9	15	·20	26		37	43	·48	54		65	71	·76	82		93	99
·4	10		21	27	·32	38		49	55	·60	66		77	83	·88	94	
5	11	·16	22		33	39	·44	50		61	67	·72	78		89	95	

Centuries before and after Christ.

·0	—	1	–	2	—	3	N. S. } B. C.
2	3	4	5	6	0	1	O. S. } B. C.
5	4	3	2	1	0	6	O. S. } A. C.
·0	3	—	2	—	1	—	N. S. } A. C.
A	G	F	E	D	C	B	Jan. Oct.
G	F	E	D	C	B	A	JAN. Apr. July.
F	E	D	C	B	A	G	Sep. Dec.
E	D	C	B	A	G	F	June.
D	C	B	A	G	F	E	Feb. Mar .Nov.
C	B	A	G	F	E	D	FEB. Aug.
B	A	G	F	E	D	C	May.
Su.	Mo.	Tu.	We.	Th.	Fr.	Sa.	1 8 15 22 29
Mo.	Tu.	We.	Th.	Fr.	Sa.	Su.	2 9 16 23 30
Tu.	We	Th.	Fr.	Sa.	Su.	Mo.	3 10 17 24 31
We	Th.	Fr.	Sa.	Su.	Mo.	Tu.	4 11 18 25
Th.	Fr.	Sa.	Su.	Mo.	Tu.	We.	5 12 19 26
Fr.	Sa.	Su.	Mo.	Tu.	We.	Th.	6 13 20 27
Sa.	Su.	Mo.	Tu.	We.	Th.	Fr.	7 14 21 28

Now, the Dominical letter of the given year being known, take it on the same horizontal line with the given month, and directly under it (viz. the Dominical letter), on the same horizontal line with the given date of the month, will be found the day of the week as required. And note, all the years, which are dotted are leap years except 100, 200, 300, N. S., A. C., and 101, 201, 301, N. S., B. C., which are common years, and for 100, 200, 300, N. S., B. C., take **0** for ·0, and in 101, 201, 301, N. S., B. C., take **1** for .1, and for Jan. and Feb. in leap years take JAN. or FEB. printed in Small Capitals.

Remaining Years above Centuries before Christ.

																			N. S.
·1	6	12		23	·29	34	40		51	·57	62	68		79	.85	90	96		**0**
0		11	·17	22	28		39	·45	50	56		67	·73	78	84		95		
	·5	10	16		27	·33	38	44		55	·61	66	72		83	·89	94		
	4		15	·21	26	32		43	·49	54	60		71	·77	82	88		99	
	3	·9	14	20		31	·37	42	48		59	·65	70	76		87	·93	98	
	2	8		19	·25	30	36		47	·53	58	64		75	·81	86	92		
		7	·13	18	24		35	·41	46	52		63	·69	74	80		91	·97	**1**

Note. A cycle of the Julian Calendar is 700 years, and that of the Gregorian Calendar 400 years; and on this principle the formulas of this Calendar are based.

458. By this table may be obtained :—

(a) The day of the week when the year, month and day of the month are given ; thus,

On what day of the week was June 17, 1857 ?

Look for the year, 57, in the left division of the table ; then pass to the right to the century column, 2, N. S., A. C., and find D for the dominical letter ; then find D in the line marked June and pass down that column until, against 17 in the days of the month, stands *Wednesday*, Ans.

(b) The days of the month which occur in any given year month and day of the week ; thus,

Thursday occurred on what days of May, 1857 ?

Having found D, the dominical letter for 1857, take D in the same line with May, and pass down the column to Thursday, against which are found the 7th, 14th, 21st and 28th, Ans.

(c) The years of a century in which any particular day of a given month, a given day of the week occurs ; thus,

In what years of the 19th century does Friday occur on the 12th of June ?

First look in the line of the 12th of the month, near the bottom of the table, and find Friday ; then ascend that column to the line of June, where D is found ; take D in the century column, 2, N. S., A. C., and at the left will be found 1, 7, 12, 18, 29, 35, 40, 46, 57, 63, 68, 74, 85, 91, 96, Ans.

(d) In what months a particular day of the month occurs on a given day of the week in a given year ; thus,

In what months of 1857 does Tuesday occur on the 3d of the month ?

Having found D to be the dominical letter for 1857, in the same line with 3, near the bottom of the table find Tuesday, ascend that column to D, the dominical letter, against which find Feb., Mar. and Nov., Ans.

459. From the above it will be seen that, in any given century, if any 3 of the 4 particulars, the year, month, day of the month and day of the week, be given, the other 1 can be found exactly or it can be limited to one of a few.

460. Ex. 1. What was the dominical letter for 1825, N. S., A. C.? $1825 \div 400 = 4$ and a remainder of 225.

Under 2 centuries N. S., A. C., and against 25 on the *left* will be found B, the dominical letter for 1825, Ans.

2. What was the dominical letter for 301, N. S., B. C.?

Under 3 centuries N. S., B. C., and against 1 on the *right* under N. S., will be found C, Ans.

3. Find the dominical letter for 1751, O. S.

$1751 \div 700$ gives a remainder of 351 and this gives F for dominical letter, A. C., and G for 1751, B. C., Ans.

4. Give the day of the week corresponding to Sept. 6, 1777, N. S., A. C.

$1777 \div 400$ gives a remainder of 177 for the tabular date. Under 1 century, N. S., A. C., and against 77 on the left is found E, the dominical letter. Under E in the line of Sept. and against the 6th day of the month, near the *bottom of the table*, is *Saturday*, the day of the week required, Ans.

5. Give the day of the week for Nov. 13, 1816, N. S., A. C.
Ans. Wednesday.

6. Also for Feb. 29, 1816, N. S., A. C. Ans. Thursday.

NOTE.—In Ex. 6, take F, the Sunday letter, in the line of FEB.

7. Give the day of the week for April 1, 1725, O. S.
Ans. Thursday, A. C., and Friday, B. C.

On what day of the week did each of the following events occur, viz.:—

8. The embarkation of Columbus from Palos, Aug. 3, 1492, O. S.? Ans. Friday.

9. The discovery of San Salvador, Oct. 12 1492?
Ans. Friday.

10. The discovery of the American continent by Cabot, July 3, 1497, O. S.? Ans. Monday.

11. The settlement of Jamestown, May 13, 1607, O. S.? Ans. Wednesday.

12. The settlement of Plymouth, Dec. 21,* 1620, N. S.? Ans. Monday.

13. The battle of Lexington, April 19, 1775? Ans. Wednesday.

14. The battle of Bunker Hill, June 17, 1775? Ans. Saturday.

15. The Declaration of Independence, July 4, 1776? Ans. Thursday.

16. The surrender of Burgoyne, Oct. 17, 1777? Ans. Friday

17. The surrender of Cornwallis, Oct. 19, 1781? Ans. Friday.

18. The Inauguration of Washington, April 30, 1789? Ans. Thursday

19. The battle of New Orleans, Jan. 8, 1815? Ans. Sunday.

20. The battle of the Nile, Aug. 1, 1798? Ans. Wednesday.

21. The battle of Cape Trafalgar, Oct. 21, 1805? Ans. Monday.

22. The battle of Waterloo, June 18, 1815. Ans. Sunday.

23. The battle of Borodino, Sept. 7, 1812? Ans. Monday.

24. The battle of Churubusco, Aug. 20, 1847? Ans. Friday.

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* The landing of the Puritans is celebrated Dec. 22d; but it is *known* that they landed on *Monday*, Dec. 11, O. S. Now, the Julian and Gregorian calendars agreed A. D. 200, and it is easy to prove that the error in the Julian calendar, in 1620, was but 10 days, and also that Dec. 21st, and not 22d, 1620, was *Monday*.

25. An elderly lady says she was born on the last Tuesday of May, 1775, N. S., A. C.; required the day of the month?

Ans. 30th.

26. In what years of the 19th century, N. S., A. C., does the 29th of Feb. fall on Thursday? Ans. 1816, 1844 and 1872.

27. In what years of the 19th century, N. S., A. C., is the 29th of Feb. on Friday? Ans. 1828, 1856 and 1884.

On what days of the week were the following births and deaths, viz.:—

28. Shakspeare, born April 23, 1564; died April 23, 1616, O. S.? Ans. b. Sunday; d. Tuesday.

29. Milton, b. Dec. 9, 1608; d. Nov. 8, 1674, O. S.?

Ans. b. Friday; d. Sunday.

30. Byron, b. Jan. 22, 1788; d. Apr. 19, 1824, N. S.?

Ans. b. Tuesday; d. Monday.

31. Whitefield, b. Dec. 16, 1714, O. S.; d. Sept. 30, 1770, N. S.? Ans. b. Thursday; d. Sunday.

32. Jonathan Edwards, b. Oct. 5, 1703; d. March 22, 1758, O. S.?

33. Sir Matthew Hale, b. Nov. 1, 1609; d. Dec. 25, 1676, O. S.?

34. Lord Bacon, b. Jan. 22, 1561; d. Apr. 9, 1626, O. S.?

35. Sir Isaac Newton, b. Dec. 25, 1642; d. Mar. 20, 1727, O. S.?

36. Martin Luther, b. Nov. 10, 1483; d. Feb. 18, 1546, O. S.?

37. Cromwell, b. Apr. 25, 1599; d. Sept. 3, 1658, O. S.?

38. Napoleon, b. Aug. 15, 1769; d. May 5, 1821, N. S.?

39. Washington, b. Feb. 11, 1731, O. S.; d. Dec. 14, 1799, N. S.?

40. Lafayette, b. Sept. 6, 1757; d. May 19, 1834, N. S.?

41. Lord Chatham, b. Nov. 15, 1708; d. May 11, 1778.

42. Dr. Franklin, b. Jan. 17, 1706; d. Apr. 17, 1790, N. S.?

43. Noah Webster, b. Oct. 16, 1758; d. May 28, 1843, N. S.?

44. Daniel Webster, b. Jan. 18, 1782; d. Oct. 24, 1852, N. S.?

45. On what day of the week were *you* born?

46. On what day of the week does your birthday occur the present year?

461. Having found on what day of the week January commences in any given year, it is easy to determine on what day of the week each month commences, by the following couplet:—

At Dover Dwells George Brown Esquire,
Good Carlos Finch And David Friar.

The initial letters of the several words represent the months in their order, the 1st word the 1st month, the 2d word the 2d month, etc. Now, if January comes in on Sunday, then D, the letter for February, being the 3d letter after A, indicates that February comes in on Wednesday, 3 days later in the week. In leap years, the months after February come in one day later.

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§ 51. APPLICATION OF ARITHMETIC TO GEOMETRY.

462. Definitions.

1. A *point* has neither length, breadth nor thickness, but *position* only.

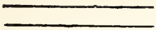
2. A *line* has *length*, but no breadth or thickness.

3. A *right line*, or *straight line*, extends only in one direction from one end of it to the other; it is also the shortest distance between two points.

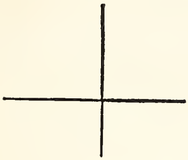
4. A *curved line* constantly changes its direction.

5. A *broken line* is composed of two or more straight lines.

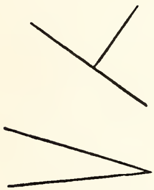
NOTE.—The word *line* alone usually signifies a *straight line;* and the word *curve,* a *curved line.*



6. Two *parallel* lines are everywhere equally distant from each other; ∴ they cannot meet if extended.



7. Two lines are *perpendicular* to each other when they meet so as to form *right angles* (329, Def. 10).



8. Two lines are *oblique* to each other when they meet so as to form *acute* or *obtuse angles* (329, Def. 10).

9. An *area, surface* or *superficies* has *length* and *breadth*, but no thickness.

10. *Surfaces* are *plane* or *curved.*

11. A *plane surface* is such that, if *any* two points are assumed upon it, the *straight* line joining the points *will lie wholly upon the surface.*

12. Two planes which are everywhere equally distant are *parallel.*

13. A *curved surface is constantly changing its direction;* as, e. g., the surface of a globe, the convex surface of a cylinder, cone, etc.

14. A *plane figure* is a plane surface bounded by straight or curved lines.

15. A *polygon* is a plane figure bounded by *straight lines.*

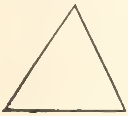
NOTE. — Three straight lines, at least, are required to bound a polygon

16. The broken line which bounds a polygon is called the *perimeter* of the polygon.

17. A polygon of 3 sides is called a *triangle;* of 4 sides, a *quadrangle* or *quadrilateral;* 5, a *pentagon;* 6, a *hexagon;* 7,

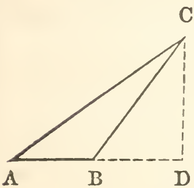
a *heptagon ;* 8, an *octagon ;* 9, a *nonagon ;* 10, a *decagon ;* 11, an *undecagon ;* 12, a *dodecagon ;* etc.

18. *Triangles,* with reference to their *sides,* are divided into *three* classes : —



1st. *Equilateral,* when the *three* sides are *equal ;*

2d. *Isosceles,* when *two* sides only are equal ; and,



3d. *Scalene,* when the three sides are *unequal,* as A B C.

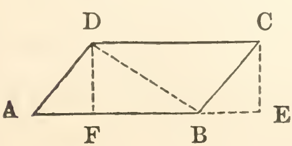
19. *Triangles,* with reference to their *angles,* are also divided into three classes : —

1st. The *right-angled triangle,* which has *one right-angle ;* (p. 220, Fig. 3).

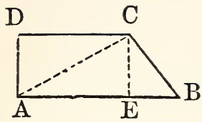
2d. The *obtuse-angled triangle,* which has *one obtuse angle,* as A B C, Def. 18 ; and,

3d. The *acute-angled triangle,* which has *three acute angles,* as Fig. 1st and 2d, Def. 18.

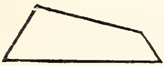
20. *Quadrangles* are of *three* kinds :—



1st. *Parallelograms,* each pair of whose opposite sides are *parallel,* as A B C D ;



2d *Trapezoids*, only *one* pair of whose sides are parallel; and,



3d. *Trapeziums*, none of whose sides are parallel.

21. *Parallelograms* are divided into *two* classes:—

1st. *Rectangles*, whose angles are all *right* (p. 221, Fig. 7); and,

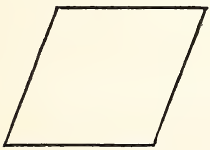
2d. *Oblique-angled parallelograms*, whose *opposite* angles are *equal*, *two* of them being *acute* and *two obtuse*, as A B C D, Fig. 1st, Def. 20.

22. *Rectangles* are of two kinds:—

1st. The *square* (p. 221, Fig. 8), which is equilateral; and,

2d. The *oblong rectangle* (p. 221, Fig. 7), whose *adjacent* sides are *unequal.*

23. *Oblique-angled parallelograms* are of *two* kinds:—



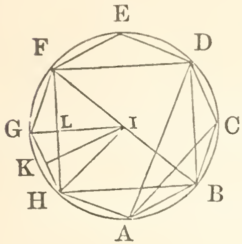
1st. *The rhombus* or *lozenge*, which is *equilateral;* and,

2d. The *rhomboid*, whose *adjacent* sides are *unequal*, as A B C D, Fig. 1st, Def. 20.

24. A *regular polygon* is one which is both *equilateral* and *equiangular.*

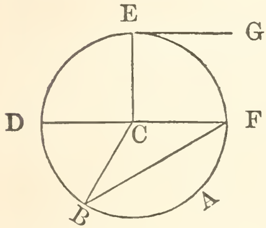
The *equilateral triangle* and the *square* are *regular polygons.*

25. *Similar polygons* have the *same number of angles*, all the angles mutually equal, and the equal angles included between *proportional sides.*



26. The *apothem* of a regular polygon is the *perpendicular* drawn from the center of the polygon to the middle of either side, as IL in the square, and IK in the octagon.

27. *The diagonal* of a polygon is a line which joins the vertices of two angles that are not adjacent, as AC or AD.



28. A *circle* is a plane figure, bounded by a curve equidistant from a point within, called the *center*.

29. The *circumference* of the circle is the *bounding curve*.

30. An *arc* is any part of the circumference, as BAF.

31. A *chord* is a line joining the extremities of an arc, as BF.

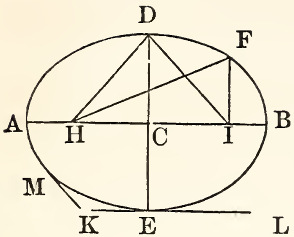
32. A *diameter* is a chord passing through the center, as DF.

33. A *radius* is the distance from the center to the circumference, as CB, CD, CE, etc.

34. A *segment* of a circle is the portion lying between an arc and its chord, as BAF.

NOTE.—When the chord is *diameter*, the *arc* is *semicircumference*, and the *segment* is *semicircle*.

35. A *sector* of a circle is the portion lying between two radii and their included arc, as BCD or DCE.



36. *An ellipse* is a plane figure, bounded by a curved line of such a form, that the sum of the distances of any point in the curve from two fixed points within, is equal to a constant quantity.

The *two fixed points* are called the *foci* of the ellipse, as H and I.

The *constant quantity* is equal to the *longest diameter* of the ellipse, viz: AB, *the diameter which passes through the foci;* thus, HD + DI = HF + FI = AB.

37. The *transverse axis* is the *longest diameter*, AB.

38. The *conjugate axis* is the *shortest diameter*, viz: ED, the diameter which is *perpendicular* to the *transverse axis.*

39. A *tangent line* is a line that *touches* a *circle, ellipse,* or *other curve* in *one point*, and which cannot touch it in any other point, however far the line may be produced or extended in either direction, as GE in the *circle*, Def. 28, and KL or KM in the *ellipse.*

40. A polygon is said to be *inscribed* in a circle or ellipse when the *vertex of each angle is in the bounding curve*, as ABCD, etc., Def. 26.

The circle or ellipse is then said to be *circumscribed* about the polygon.

41. A polygon is said to be *circumscribed* about a circle or ellipse, when *each side of the polygon is tangent to the curve*, as ABCD, Fig. 10, p. 222.

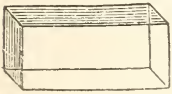
The circle or ellipse is then said to be *inscribed* in the polygon.

NOTE.—Mathematical points, lines, and surfaces, exist only in imagination, but we use representations of them to aid us in mathematical investigations.

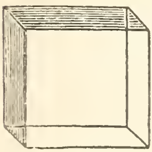
42. A *solid* or *body* is a figure which has *length, breadth, and thickness.*

43. A *prism* is a solid that has two similar, equal, parallel faces, called *bases.* and all its other faces parallelograms.

NOTE.—A prism is triangular, quadrangular, pentagonal, etc., according as its bases are triangles, quadrangles, pentagons, etc.



44. A *parallelopipedon* is a prism bounded by six parallelograms.



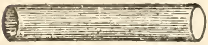
If these six parallelograms are *rectangles,* the parallelopipedon is *rectangular;* if they are *equal* rectangles, it is a *cube.*

45. A *pyramid* is a solid, having a polygonal face, called the *base,* and all its other faces are triangles which meet at a common point, called the *vertex* of the pyramid.

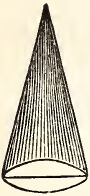
46. A *right pyramid* is one whose base is a *regular* polygon, and in which the perpendicular let fall from the vertex to the base, passes through the *center* of the base.

47. An *oblique pyramid* is one in which the perpendicular, from the vertex to the base, does *not* pass through the *center* of the base.

NOTE.—A pyramid is triangular, quadrangular, etc., according as its base is a triangle, quadrangle, etc.

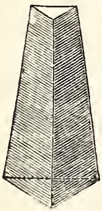


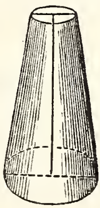
48. A *cylinder* is a round body whose diameter is the same throughout its entire length, and whose ends or bases are equal, parallel circles.



49. A *cone* is a solid, like a pyramid, except its *base* is *a circle.*

NOTE.—The cone is *right* or *oblique* in like manner with the pyramid.





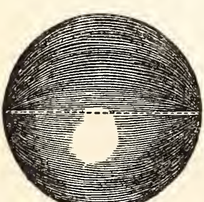
50. The *frustum of a pyramid* or *cone* is the part remaining after a portion next the vertex has been cut off by a plane parallel to the base.

51. A *wedge* is a solid bounded by five plane faces, one of which, called the *back,* is a quadrangle, and usually a rectangle, two of them, called the *ends*, are triangles, and the other two, called the *sides,* are trapezoids or parallelograms.

The line in which the sides meet is called the *edge* of the wedge.

NOTE.—*A right wedge* has its back and sides *rectangles,* and ∴ its ends are parallel to each other, and perpendicular to the back.

52. A *rectangular prismoid* is a solid resembling the frustum of a rectangular pyramid, but differing from it in this, that the trapezoids forming its sides, if extended, would not meet in a common vertex, but one pair of them would meet before the other, and thus form a *wedge* instead of a pyramid.



53. A *sphere* is a solid bounded by a curved surface, all parts of the surface being equally distant from a point within, called the *center.*

54. A *diameter* of the sphere is a line passing through the center, and limited in both directions by the surface.

NOTE.—All diameters of the same sphere are equal.

55. A *radius* or *semidiameter* of a sphere is the distance from the center to the surface.

NOTE 1.—All radii of the same sphere are equal.

NOTE 2.—If a plane be passed through a sphere, the section so made will be a *circle*. If the plane passes through the *center*, the section is a *great circle*; if it does *not* pass through the center, the section is a *small circle*.

56. A *spherical segment* is the portion of the sphere cut off by a plane, or the portion lying between two *parallel* secant or cutting planes.

NOTE 1.—The sections formed by the secant planes are the *bases* of the segments.

NOTE 2.—*A Hemisphere* is the segment cut off by a secant plane passing through the center.

57. A *zone* is that portion of the *surface* of a sphere cut off by a secant plane, or the portion included between two parallel secant planes.

58. If two *great circles* intersect each other upon the surface of a sphere, they *bisect* each other; i. e. they divide each other into two *equal* parts; they also have a *common diameter*.

59. The portion of the *surface* included between either pair of semicircles is called a *lune*, and the portion of the sphere cut out by these two semicircles is called an *ungula* or *spherical wedge*.

60. If *three* arcs of *great circles* enclose a portion of the surface of a sphere, that portion is a *spherical triangle;* if *four* or *more* arcs enclose a portion of the surface, such portion is a *spherical polygon*.

61. If planes be passed through the arcs of a spherical triangle or polygon, *they will pass through the center of the sphere* and form a solid angle at the center. The portion of the sphere lying between these planes is a *spherical pyramid*.

62. A *spherical sector* is a portion of a sphere composed of *a spherical segment and a cone having the same base as the segment, and its vertex at the center of the sphere*.

NOTE 1.—The segment, in this case, must have but *one base*

NOTE 2.—If the sector is more than a hemisphere *the cone must be subtracted from the segment instead of being added to it.*

NOTE 3.—If a smaller sector, concentric with a larger, be taken out of the larger, *the portion of the larger sector remaining, is also called a sector.*

63. The *base* of a figure is the side on which it is supposed to stand.

NOTE.—Most figures are said to have *two* bases, *an upper and a lower base.*

64. The *altitude* of a figure is its *perpendicular hight.*

(a) The altitude of a *triangle* is the perpendicular let fall upon the base from the vertex of the opposite angle, as C D, Fig. 11, p. 222, or upon the base produced, as C D, 3d Fig., Def. 18.

(b) The altitude of a *parallelogram* is the *perpendicular* distance between *either pair of its parallel sides taken as bases,* as C E, 1st Fig., Def. 20.

(c) The altitude of a trapezoid is the perpendicular distance between its *parallel* sides, as D A, 2d Fig., Def. 20.

(d) The altitude of a *pyramid* or *cone* is the perpendicular distance *from its vertex to its base* or *base produced.*

(e) The altitude of a *prism, cylinder, prismoid, frustum of a pyramid or cone, spherical segment* or *zone* is the *perpendicular* distance between its *parallel bases.*

(f) The altitude of a *wedge* is the *perpendicular* distance from its *edge* to its *back.*

64. The *slant hight* of a *right pyramid* is the *perpendicular drawn from the vertex to either side of the polygon which forms the base.*

65. The *slant hight* of a *right cone* is the distance from the *vertex to any point in the circumference of the base.*

66. The *slant hight* of the *frustum of a right pyramid* is the perpendicular distance between the corresponding edges of the two parallel bases.

67. The *slant hight* of the *frustum of a right cone is the shortest distance between the circumferences of its two bases.*

MENSURATION OF SURFACES AND SOLIDS.

463. The following are the problems in most frequent use in Mensuration, and the rules given for their solution are all

easily proved by Geometrical reasoning. However, no attempt is made to *prove* them in this work, though some of them are familiarly explained.

464. The *approximate* ratio of the circumference of a circle to its diameter, 3.141592, is indelibly fixed in the mind of every Geometer, and for this reason free use of it is made in the following rules; for a like reason, rules depending on obvious and easily remembered principles are given in preference to others more brief, deduced from Algebraic formulas, which require the recollection of many and extended decimals. These shorter methods, however, are frequently given in a second or third rule.

465. Prob. 1.—To find the area of a parallelogram.

Rule.—*Multiply the base by the altitude.*

Ex. 1. What is the area of a parallelogram whose base is 10 inches and altitude 4 inches? Ans. 40 sq. inches.

The reason is obvious in the *rectangle* A B C D, Art. 75. It is also apparent in Fig. 1st, Def. 20, that the triangle B C E, which is in the rectangle F E C D, but not in the rhomboid, A B C D is equal to the triangle A F D, which is in the rhomboid but out of the rectangle; ∴ the rectangle and the rhomboid are equal; i. e., *all parallelograms* that have equal bases and equal altitudes have also equal areas, whether they are *rectangular* or *oblique-angled; hence the rule is universal.*

2. What is the area of a parallelogram whose base is 20 rods and altitude 7 rods? Ans. 140 sq. rd.

3. What is the area of a parallelogram whose base is 2 feet and altitude 3 inches? Ans. 72 sq. in.

4. What is the area of a rhomboidal piece of land whose adjacent sides are respectively 87 and 35 rods, provided the perpendicular let fall from the vertex of one angle meets the opposite base 21 rods from the adjacent angle at that base, assuming the longer sides for bases? Ans. 2436 sq. rd.

466. Prob. 2.—To find the area of a triangle.

RULE.—*Multiply the base by half the altitude, or half the base by the altitude.*

Ex. 1. What is the area of a triangle whose base is 10 inches and altitude 4 inches? Ans. 20 sq. in.

The diagonal of a parallelogram divides it into two equal triangles whose bases and altitudes are respectively equal to the base and altitude of the parallelogram, Fig. 1st, Def. 20; i. e., *the area of a triangle is one half the area of a parallelogram having the same base and altitude;* hence the rule.

2. What is the area of a triangle whose base is 20 feet and altitude 7 feet? Ans. 70 sq. ft.

3. What is the area of a triangle whose base is 2 yards and altitude 3 feet? Ans. 9 sq. ft.

4. What is the area of a triangle whose base is 75 feet and another side 35 feet, provided the perpendicular from the vertex of the triangle meets the base or the base produced 28 feet from the angle formed by the two given sides? Ans. $787\frac{1}{2}$ sq. ft.

467. When the three sides of a triangle are given, its area may be found by the following

RULE.—*From the half sum of the three sides, subtract each side separately; multiply together the half sum and three remainders, and the square root of the continued product will be the area sought.*

5. The sides of a triangle are 6, 8 and 10 feet; what is its area?

$(6+8+10)\div 2=12$; $12-6=6$; $12-8=4$; $12-10=2$; $\sqrt{12\times 6\times 4\times 2}=24$. Ans. 24 sq. ft.

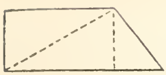
6. The sides of a triangle are, 7, 12 and 15; what is its area?

Ans. $\sqrt{1700}=41.23+$.

468. PROB. 3.—To find the area of a trapezoid,

RULE.—*Multiply the half sum of the parallel sides by the altitude.*

Ex. 1. The parallel sides of a trapezoid are 7 and 11 feet and its altitude 4 feet; what is its area? Ans. 36sq. ft.



The diagonal of a trapezoid divides it into two triangles whose bases are the parallel sides of the trapezoid and whose *common* altitude is the altitude of the trapezoid; hence the rule.

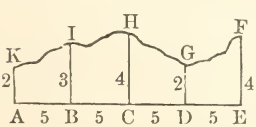
2. What is the area of a trapezoid whose altitude is 6 and whose parallel sides are 10 and 15? Ans. 75.

469. One important application of this Prob. is the measuring of a long irregular piece of land, bounded on one side by a straight line, as, e. g., a strip lying between a railroad and a river, running nearly parallel.

This may be done by the following

RULE.—*Measure the long, straight bounding line, divide it into a convenient number of equal parts, and at the points of division and extremities of the line erect perpendiculars and extend them to the opposite side of the lot; then to the half sum of the extreme perpendiculars add the sum of all the intermediate perpendiculars, multiply this last sum by one of the equal divisions of the straight line and the product will be the area, nearly.*

Ex. 3. There is a piece of land 20 rods long, 2 rods wide at one end and 4 rods at the other, and at 3 intermediate points equally distant from each other it is 3, 4 and 2 rods wide as represented in the figure; what is its area?



$20 \div 4 = 5$, length of one division.
$(\frac{2}{2} + 3 + 4 + 2 + \frac{4}{2}) \times 5 = 60$ sq. rd. Ans.

This is upon the supposition that the irregular lot is divided into trapezoids, the perpendiculars being the parallel sides of the trapezoids and one of the equal divisions of the straight bounding line their *common* altitude; thus, the measure of A B I K $= \frac{2+3}{2} \times 5 = 12.5$; B C H I $= \frac{3+4}{2} \times 5 = 17.5$;

$CDGH = \frac{4+2}{2} \times 5 = 15$; and $DEFG = \frac{2+4}{2} \times 5 = 15$; $\therefore$ the sum of the four trapezoids, i. e. the entire lot $= 12.5 + 17.5 + 15 + 15 = 60$ sq. rods; or it may be represented as follows :—$\left(\frac{2+3}{2} + \frac{3+4}{2} + \frac{4+2}{2} + \frac{2+4}{2}\right) \times 5 = \left(\frac{2}{2} + 3 + 4 + 2 + \frac{4}{2}\right) \times 5 = 60$sq. rods, as before.

4. There is a strip of land 24 rods long, 3 rods wide at one end and 5 rods at the other, and at 5 equidistant intermediate points it is 6, 4, 7, 5 and 3 rods wide; what is its area?

Ans. 116sq. rods.

NOTE 1.—If the strip comes to a point at one or both ends, then one or each extreme perpendicular becomes zero and the rule is equally applicable, for then one or each extreme trapezoid becomes a triangle.

NOTE 2.—If the form of the land is such as to make it more convenient, the perpendiculars may be erected at *unequal* distances from each other, the trapezoids and triangles calculated separately, and the sum of their areas will be the entire area of the lot.

470. PROB. 4.—To find the area of any regular polygon,

RULE 1.—*Multiply the perimeter of the polygon by half its apothem.*

Ex. 1. One side of a regular hexagon is 10 feet and its apothem is 8.660254 feet, nearly; what is its area?

$6 \times 10 \times \frac{8.660254}{2} = 259.80762$sq. ft., Ans.

If lines be drawn from the centre to the vertices of all the angles the polygon will be divided into as many equal triangles as it has sides, the sides of the polygon being the bases of the triangles and the apothem of the polygon being their *common* altitude; hence the rule.

2. Each side of a square is 10 feet and its apothem 5 feet, what is its area? Ans. 100sq. ft.

3. A side of a regular octagon is 10 and its apothem 12.071068; what is its area? Ans. 482.84272.

471. The areas of all similar figures are to each other as the squares of their homologous sides (330, 9), ∴ we may more readily obtain the areas of regular polygons by reference to a table in which the areas of the regular polygons are given when each side is a unit.

The following is such a

TABLE.

Name.	No. Sides.	Apothem.	Area.
Triangle,	3	0.2886751	0.4330127
Square,	4	0.5000000	1.0000000
Pentagon,	5	0.6881910	1.7204774
Hexagon,	6	0.8660254	2.5980762
Heptagon,	7	1.0382607	3.6339124
Octagon,	8	1.2071068	4.8284271
Nonagon,	9	1.3737387	6.1818242
Decagon,	10	1.5388418	7.6942088
Undecagon,	11	1.7028436	9.3656399
Dodecagon,	12	1.8660254	1. 1961524

RULE 2.—*Square one side of the polygon whose area is required, multiply this square by the tabular area of the polygon having the same number of sides, and the product will be the area sought.*

Ex. 4. What is the area of a regular triangle, square, hexagon and dodecagon, one side of each being 10 inches?

$$\left.\begin{array}{lllr} & & & \text{sq. in.} \\ \text{Triangle} & = 0.4330127 & \times 10^2 = & 43.30127 \\ \text{Square} & = 1. & \times 10^2 = & 100. \\ \text{Hexagon} & = 2.5980762 & \times 10^2 = & 259.80762 \\ \text{Dodecagon} & = 11.1961524 & \times 10^2 = & 1119.61524 \end{array}\right\} \text{Ans.}$$

5. What is the area of a regular octagon, one of whose sides is 5 rods? Ans. 120.7106775 sq. rods.

6. What is the area of a regular dodecagon whose side is 8? Ans. 716.5537536

472. PROB. 5. — To find the area of a trapezium or of any irregular polygon of more than 4 sides,

RULE. — *Divide the polygon into triangles by drawing diagonals from the vertex of one angle to the vertices of all the opposite angles, and the sum of the areas of these triangles will be the area sought.*

473. PROB. 6. — To find the circumference of a circle when the diameter is given,

RULE. — *Multiply the diameter by* 3.141592, *and the product will be the circumference, nearly.*

Ex. 1. What is the circumference of a circle whose diameter is 10 miles? Ans. 31.41592 miles.

2. What is the circumference of a circle whose diameter is 25 feet? Ans. 78.5398 feet.

474. PROB. 7. — To find the length of an arc of a circle containing a given number of degrees,

RULE. — *Find the length of the whole circumference, and then say, as* 360° *is to the number of degrees in the arc, so is the length of the circumference to the length of the arc.*

Ex. 1. What is the length of an arc of 45° in a circle whose diameter is 10 inches? Ans. 3.92699 inches.

2. What is the length of an arc of 30° in a circle whose radius is $12\frac{1}{2}$ feet? Ans. $6.5449\frac{5}{6}$ feet.

475. PROB. 8. — To find the diameter of a circle when the circumference is given,

RULE. — *Divide the circumference by* 3.141592, *and the quotient will be the diameter, nearly.*

Ex. 1. What is the diameter of a circle whose circumference is 314.1592 inches? Ans. 100 inches.

2. What is the diameter of a circle whose circumference is 87.964576 barleycorns? Ans. 28b.c.

476. Prob. 9. — To find the area of a circle when its diameter is given,

Rule 1. — *First find the circumference (Prob. 6.), and then multiply the circumference by half the radius.*

This rule is founded in the hypothesis that a circle is made up of an infinite number of triangles whose vertices are at the center, and whose bases together constitute the circumference; thus, in the circle, Def. 26, first inscribe a square B D F H, and then a regular octagon A B C D E F G H, and we readily see that the area of the octagon is nearer like the area of the circle than that of the square is, and the perimeter of the octagon is more nearly equal to the circumference of the circle than the perimeter of the square is; so, also, if the square and octagon be divided into triangles by drawing lines from the center to the vertices of the angles, the altitude I K of the triangles which compose the octagon is more nearly equal to the radius of the circle than the altitude I L of the triangles that compose the square is; and the more sides there are to the inscribed polygon, the more nearly will the altitude, the sum of the bases and the sum of the areas of the triangles, respectively, approach to the radius, the circumference and the area of the circle; hence the correctness of the rule.

Ex. 1. What is the area of a circle whose diameter is unity?

Since the diameter is 1, the circumference is 3.141592 (330, 3); $\therefore 3.141592 \times \frac{1}{4} = .785398$, Ans.

2. What is the area of a circle whose radius is unity?

$3.141592 \times 2 = 6.283184 =$ circumference; $6.283184 \times \frac{1}{2} = 3.141592 =$ area, Ans.; i. e., when the radius is unity, the area of the circle is expressed by the same figures that indicate the ratio of the circumference to the diameter.

477. Now, since the areas of circles are to each other as the squares of their diameters, or as the squares of their radii (330, 5), we have from the two preceding examples,

To find the area of a circle,

Rule 2. — *Multiply the square of the diameter by the decimal* .785398 ; *or,*

Rule 3. — *Multiply the square of the radius by* 3.141592.

Ex. 3. What is the area of a circle whose diameter is 10 inches? Ans. 78.5398 sq. in.

4. What is the area of a circle whose diameter is 5 rods? Ans. 19.63495 sq. rd.

5. What is the area of a circle whose radius is 100 miles? Ans. 31415.92 sq. miles.

6. What is the area of a circle whose radius is 25 yards? Ans. 1963.495 sq. yards.

478. Prob. 10. — To find the area of a circular sector,

Rule 1. — *First find the area of the whole circle, and then say, as* 360° *is to the number of degrees in the arc of the sector, so is the area of the circle to the area of the sector ; or,*

Rule 2. — *First find the length of the arc of the sector, and then multiply this length by* $\frac{1}{2}$ *the radius.*

Ex. 1. What is the area of a circular sector whose arc is 90°, in a circle whose diameter is 10 feet? Ans. 19.63495 sq. ft.

2. What is the area of a sector whose arc is 120°, in a circle whose radius is 100 miles? Ans. 10471.97$\frac{1}{3}$ sq. m.

479. Prob. 11. — To find the area of a circular segment,

Rule. — *Find the area of a sector having the same arc as the segment, also of a triangle formed by the chord of the arc and the two radii of the sector, and then take the triangle from the sector if the segment is less than semicircle, and add the two together if it is greater.*

Ex. 1. What is the area of a circular segment whose arc is 120° in a circle whose radius is 100 miles? Ans. 6141.846$\frac{1}{3}$ sq. m.

2. What is the area of a segment whose arc is 60° in a circle whose radius is 100 miles? Ans. 905.859$\frac{2}{3}$ sq. m.

480. Prob. 12. — To find the area of a circle when its circumference is given,

Rule. — *Find the diameter by Prob. 8, and the area by Prob. 9.*

Ex. 1. What is the area of a circle whose circumference is 314.1592?

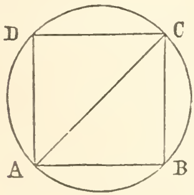
$$314.1592 \div 3.141592 = 100, \text{ the diameter.}$$
$$.785398 \times 100^2 = 7853.98, \text{ the area, Ans.}$$

2. What is the side of a square whose area is equal to that of a circle whose circumference is 87.964576 miles?

Ans. 24.814 miles.

481. Prob. 13.—The diameter of a circle being given to find the side of an inscribed square,

Rule.—*Extract the square root of half of the square of the diameter.*



The diameter A C divides the inscribed square into two equal right-angled, isosceles triangles, of which A C is the hypothenuse, $\therefore AC^2 = AB^2 + BC^2 = 2\,AB^2$, $\therefore AB^2 = \frac{AC^2}{2}$ and $AB = \sqrt{\frac{AC^2}{2}}$ and this is the enunciation of the rule.

Ex. 1. The diameter of a circle is 20 inches; what is the side of the inscribed square?

$20^2 = 400$; $400 \div 2 = 200$; and $\sqrt{200} = 14.142$ inches, Ans.

2. The diameter of a circle is 25; what is the side of the inscribed square? $\sqrt{\frac{25^2}{2}} = \sqrt{312.5} = 17.677$, Ans.

3. What is the side of the greatest square stick of timber that can be hewn from a cylindrical log 40 inches in diameter?

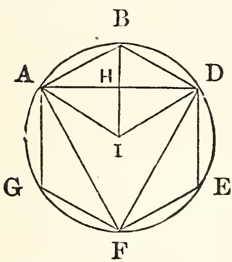
Ans. 28.284 inches.

4. The circumference of a circle is 314.1592; what is the side of the inscribed square? Ans. 70.71.

482. Prob. 14.—Having given the diameter of a circle to

find the side of a regular inscribed hexagon and of an equilateral triangle,

Rule 1.—*The radius of the circle is equal to the side of the hexagon and the square root of three times the square of the radius is equal to the side of the triangle.*



The radius of the circle being equal to the side of the hexagon, A B D I, is a rhombus, and A B, B D, D I, and I A are each equal to I B. Now it is easily proved by Geometry that the sum of the squares of the four sides of a parallelogram is equal to the sum of the squares of its two diagonals; $\therefore$ $AB^2 + BD^2 + DI^2 + IA^2 = IB^2 + AD^2$; i. e. $4\,AB^2$ or $4\,IB^2 = IB^2 + AD^2$; $\therefore$ $3\,IB^2 = AD^2$ and $AD = \sqrt{3\,IB^2}$, which agrees with the enunciation of the rule for finding the side of the inscribed equilateral triangle.

Ex. 1. What is the side of the regular inscribed hexagon and of the equilateral triangle, the diameter of the circle being 10 feet?

$$\left.\begin{array}{l}\text{Side of hexagon} = 5 \text{ feet,}\\ \text{Side of triangle} = \sqrt{3 \times 5^2} = \sqrt{75} \text{ ft.}\end{array}\right\} \text{Ans.}$$

Again, if $AI = 1$ then $IH = .5$ and $AH = \sqrt{AI^2 - IH^2} = \sqrt{1^2 - .5^2} = \sqrt{.75} = .8660254$ and $AD = 1.7320508$; $\therefore$, since all like lines of similar figures are proportional (330, 9), we may find the side of the inscribed equilateral triangle by

Rule 2.—*Multiply the diameter of the circle by the decimal* .8660254; *or,*

Rule 3.—*Multiply the radius of the circle by* 1.7320508.

Ex. 2. A round log is 24 inches in diameter; what is the side of the greatest equilateral triangular beam that may be sawn from it? Ans. 20.7846 in.

3. The circumference of a circle is 87.964576; what is the side of the inscribed regular hexagon and triangle?

$$\left.\begin{array}{l}\text{Hexagon } 14. \\ \text{Triangle } 24.2487112,\end{array}\right\}\text{Ans.}$$

483. Prob. 15.—To find the area of an ellipse,

Rule.—*Multiply the product of the semi-axes by* 3.141592.

Note.—The proof of this rule is found in Conic Sections; however, it is perfectly analogous to the 3d rule, Prob. 9, for finding the area of a circle; for the distance from the center of the ellipse to either focus is the eccentricity of the ellipse, and as this eccentricity becomes less, the ellipse more nearly approaches a circular form until, when the eccentricity becomes 0, the ellipse becomes a circle, *and then the product of the semi-axes becomes the square of the radius.*

Ex. 1. The transverse axis of an ellipse is 20 inches and the conjugate axis 16 inches; what is its area?

$\frac{20}{2} \times \frac{16}{2} \times 3.141592 = 251.32736$ sq. inches, Ans.

2. The axes of an ellipse are 30 and 18, what is its area?
Ans. 424.11492.

3. The semi-axes of an ellipse are 12 and 10, what is its area?
Ans. 376.99104.

484. Prob. 16.—To find the convex surface of a right prism or of a right cylinder,

Rule.—*Multiply the perimeter or circumference of the base by the altitude of the solid.*

Note 1.—This rule is evidently true for the prism, from Prob. 1; and it is equally true for the cylinder, since the convex surface of the cylinder is made up of an infinite number of rectangles.

Note 2.—If the *entire* surface of the solid is required, the area of the two bases must be added to the convex surface.

Ex. 1. What is the convex surface of a right prism whose altitude is 40 inches, and the perimeter of whose base is 60 inches? Ans. 2400 sq. in.

2. What is the entire surface of a right prism whose altitude

is 10 and whose base is a regular hexagon, having unity for its side? Ans. 65.1961524.

3. What is the convex surface of a right cylinder whose altitude is 12 inches and the circumference of whose base is 18 inches? Ans. 216 sq. in.

4. What is the entire surface of a right cylinder whose altitude is 25 inches and the radius of whose base is 5 inches?

Ans. 942.4776 sq. in.

5. What is the convex surface, what the surface of the two bases, and what the entire surface of a right cylinder whose altitude and diameter is each 4 feet?

$$\left.\begin{array}{ll} & \text{sq. ft.} \\ \text{Convex surface} & = 50.265472 \\ \text{Surface of two bases} & = \underline{25.132736} \\ \text{Entire surface} & = 75.398208 \end{array}\right\} \text{Ans.}$$

REMARK.—The cylinder in Ex. 5 will evidently *circumscribe* a sphere of 4 feet diameter and the *base* of the cylinder is equal to a *great circle* of this sphere. It will also be observed from the answer to Ex. 5 that the surface of the two bases is just $\frac{1}{2}$ the convex surface of the cylinder, and ∴ that the surface of one base is $\frac{1}{4}$ the convex surface of the cylinder; hence the entire surface of the cylinder is just 6 great circles of the inscribed sphere. It is also easily proved that the surface of the sphere is equal to 4 great circles; hence *the surface of the sphere is equal to the convex surface of the circumscribing cylinder* and the surface of the sphere is to the *entire* surface of the cylinder as 4 to 6, i. e., as 2 to 3. The solidity of the sphere is also to the solidity of the cylinder in the same ratio, viz., 2 to 3.

N. B. The learner will observe that only those cylinders whose diameters and altitudes are equal, can have inscribed spheres; hence these relations of surfaces and of solidities hold only in such cylinders.

185. PROB. 17.—To find the convex surface of a right pyramid or of a right cone,

RULE.—*Multiply the perimeter or circumference of the base by half the slant hight of the solid.*

NOTE 1.—If the entire surface is desired, add the area of the base

NOTE 2.—This is but an application of Prob. 2. The convex surface of the cone is made up of an infinite number of triangles whose vertices are at the apex of the cone, and whose bases make up the circumference of the base of the cone.

Ex. 1. What is the convex surface of a right pyramid whose slant hight is 18 inches and the perimeter of whose base is 27 inches? Ans. 243 sq. in.

2. What is the entire surface of a right octagonal pyramid whose slant hight is 20 feet, and each side of whose base is 2 feet? Ans. 179.3137084 sq. ft.

3. What is the convex surface of a right cone whose slant hight is 30 yards and the circumference of whose base is 24 yards? Ans. 360 sq. yd.

4. What is the entire surface of a right cone whose slant hight is 60 inches and the radius of whose base is 50 inches?

Ans. 17278.756 sq. in.

486. PROB. 18.—To find the convex surface of the frustum of a right pyramid or of a right cone,

RULE.—*Multiply the half sum of the perimeters or circumferences of the two bases by the slant hight of the solid.*

NOTE 1.—If the entire surface is wanted, add the areas of the two bases.

NOTE 2.—This rule is an application of Prob. 3. The convex surface of the frustum of a cone is composed of an infinite number of trapezoids whose longer bases make up the circumference of the lower base of the frustum and whose shorter bases make up the circumference of the upper base.

Ex. 1. What is the convex surface of the frustum of a right pyramid whose slant hight is 6 feet and the perimeters of whose bases are 5 and 15 feet? Ans. 60 sq. ft.

2. What is the entire surface of the frustum of a right pentagonal pyramid, one side of the lower base being 4, one side of the upper base 2, and the slant hight 7?

Ans. 139.409548.

3. What is the convex surface of the frustum of a right cone whose slant hight is 27 inches and the circumferences of whose bases are 33 and 27 inches? Ans. 810 sq. in.

4. What is the entire surface of the frustum of a right cone whose slant hight is 25, the circumference of whose lower base is 314.1592 and the radius of the upper base 40?

Ans. 19949.1092.

487. PROB. 19.—To find the surface of a sphere,

RULE.—*Multiply the circumference by the diameter.*

NOTE.—The surface of a sphere equals 4 great circles of the same sphere (Prob. 16, Remark), and, as we find the area of a circle by multiplying the circumference by $\frac{1}{2}$ radius or $\frac{1}{4}$ diameter (Prob. 9, Rule 1), so we find the surface of the sphere by the above rule.

Ex. 1. What is the surface of a sphere whose radius is 50 inches? Ans. 31415.92 sq. in.

2. What is the surface of a sphere whose circumference is 87.964576 feet? Ans. 2463.008128 sq. ft.

3. Suppose our earth to be a sphere whose radius is 4000 miles, what is its surface? Ans. 201,061,888 sq. miles.

4. What is the surface of the sun, supposing it to be a sphere whose diameter is 896000 miles?

Ans. 2522120323072 sq. miles.

488. PROB. 20.—To find the area of a spherical zone,

RULE.—*Multiply the circumference of a great circle by the altitude of the zone.*

Ex. 1. What is the area of a zone whose altitude is 10 inches, the radius of the sphere being 50 inches?

Ans. 3141.592 sq. in.

2. Suppose the circumference of the earth is 25132.736 miles, and that the altitude of the torrid zone is 3186 miles, what is the area of that zone? Ans. 80072896.896 sq. miles.

3. What is the area of the north temperate zone of the earth, it being 2076 miles in altitude? Ans. 52175559.936 miles.

4. What is the area of the north frigid zone of the earth, its altitude being 331 miles? Ans. 8318935.616 miles.

Do the results in the 3 last examples correspond with the result in Ex. 3, Prob. 19?

489. Prob. 21.—To find the area of a lune,

Rule.—*First find the surface of the sphere, and then say as* 360° *is to the number of degrees in the angle of the lune, so is the surface of the sphere to the surface of the lune.*

Ex. 1. What is the area of a lune whose angle is 36° on a sphere whose radius is 4000 miles?

Ans. 20106188.8 sq. miles.

2. What is the area of a lune whose angle is 18°, the diameter of the sphere being 896000? Ans. 126106016153.6.

490. Prob. 22.—To find the area of a spherical triangle,

Rule.—*Having found the surface of the sphere, add together the three angles of the triangle; from their sum subtract* 180°, *divide the remainder by* 90°, *and multiply* $\frac{1}{8}$ *of the surface of the sphere by the quotient.*

Ex. 1. What is the area of a triangle whose angles are 80°, 90°, and 130°, on a sphere whose radius is 4000 miles?

Surface of sphere = 201061888 (Prob. 19); $\frac{1}{8}$ surface of sphere = 25132736.

$$80° + 90° + 130° - 180° = 120°; \ \frac{120°}{90°} = \frac{4}{3}$$

$$25132736 \times \frac{4}{3} = 33510314\,\tfrac{2}{3} \text{ sq. miles, Ans.}$$

Note.—The sum of all the angles of a *plane* triangle is always 2 right angles, i. e. 180°, but the sum of all the angles of a *spherical* triangle is any quantity more than 2 and less than 6 right angles.

2. The angles of a triangle are 100°, 150°, and 110°, on a sphere whose diameter is 10 inches; what is its area?

Ans. 78.5398 sq. in.

491. PROB. 23.—To find the area of a spherical polygon,

RULE.—*From the sum of all the angles of the polygon subtract 180° as many times less two, as there are sides to the polygon; divide the remainder by 90°, and multiply $\frac{1}{8}$ the surface of the sphere by the quotient.*

Ex. 1. On a sphere whose radius is 50 feet, the angles of a hexagon are 100°, 105°, 125°, 140°, 150°, and 160°; what is the area of the hexagon?

$\frac{1}{8}$ surface of sphere $= 3926.99$

$100° + 105° + 125° + 140° + 150° + 160° - 180° \times 4 = 60°$; $60° \div 90° = \frac{2}{3}$;

$3926.99 \times \frac{2}{3} = 2617.99\frac{1}{3}$ sq. feet, Ans.

2. What is the area of a pentagon on a sphere whose circumference is 15.70796, the angles being 150°, 119° 30′, 75°, 145°, and 170° 30′?

Ans. $13.0899\frac{2}{3}$.

492. PROB. 24.—To find the solid contents of a prism or of a cylinder,

RULE.—*Multiply the area of the base by the altitude.*

NOTE.—It is easily proved that the solidity of every prism or cylinder is equivalent to that of a rectangular parallelopipedon having an equivalent base, and the same altitude (77); hence the rule.

Ex. 1. What is the solidity of a prism whose base is 16 square feet, and whose altitude is 8 feet?

Ans. 128 sol. ft.

2. What is the solidity of a prism whose altitude is 20 inches, and whose base is a regular nonagon, having 3 inches for its side?

Ans. 1112.728356 sol. in.

3. What are the solid contents of a cylinder whose altitude is 25 feet, and the radius of whose base is 6 feet?

Ans. 2827.4328 sol. ft.

4. What is the solidity of a cylinder whose altitude is 2 feet, and the diameter of whose base is 10 inches?

Ans. 1884.9552 sol. in.

493. PROB. 25.—To find the solidity of a pyramid or of a cone,

RULE.—*Multiply the area of the base by $\frac{1}{3}$ of the altitude.*

NOTE.—The solid contents of a pyramid or cone are found by Geometry to be just one-third the contents of a prism or cylinder having the same base and altitude.

Ex. 1. What is the solidity of a pyramid whose altitude is 27 inches, and whose base is 56 square inches?

Ans. 504 sol. in.

2. What are the solid contents of a pyramid whose altitude is 3 feet, and whose base is a regular dodecagon, having ten inches for its side? Ans. 13435.38288 sol. in.

3. What is the solidity of a cone whose altitude is 12 feet, and the radius of whose base is 12 inches?

Ans. 21714.683904 sol. in

4. What is the solidity of a right cone, whose base is 16 inches in diameter, and whose slant hight is 10 inches?

Ans. 402.123776 sol. in.

494. PROB. 26.—To find the solidity of the frustum of a pyramid or of a cone.

RULE.—*Add the areas of the two bases and their mean proportional* (257, a) *together, multiply this sum by $\frac{1}{3}$ of the altitude, and the product will be the solidity.*

NOTE —This rule is founded on the Geometrical principle that the solidity of the frustum of a pyramid or of a cone is equivalent to the solidity of *three* pyramids or cones, having a common altitude with the frustum, and for bases, the lower base of the frustum, the upper base of the frustum and a mean proportional between these bases.

Ex. 1. What is the solidity of the frustum of a pyramid whose altitude is 12 inches, and whose bases are the one 16 and the other 9 inches square?

$$16^2 = 256, \text{ lower base}$$
$$9^2 = 81, \text{ upper base}$$
$$\sqrt{256 \times 81} = \underline{144}, \text{ mean between the bases.}$$
$$481 \times 4 = 1924 \text{ sol. in., Ans.}$$

2. The altitude of the frustum of a pyramid is 18 feet, the area of its lower base 81 feet, and of its upper base 36 feet; what is its solidity? Ans. 1026 sol. ft.

3. The altitude of the frustum of a cone is 30 inches, and the areas of its bases are 225 and 64 feet; what is its solidity?

Ans. 588960 sol. in.

4. The altitude of the frustum of a cone is 15, the radius of its lower base is 5, and the circumference of its upper base is 18.849552; what are its solid contents? Ans. 769.69004.

495. Prob. 27.—To find the solidity of a wedge,

Rule.—*To twice the length of the back, add the length of the edge; multiply this sum by the breadth of the back, and this product by* $\frac{1}{6}$ *of the altitude; the last product will be the contents of the wedge.*

Ex. 1. The back of a wedge is 30 inches in length by 20 inches in breadth; the edge is 25 inches, and the altitude 36 inches; what is its solidity?

$$(30 \times 2 + 25) \times 20 \times \tfrac{36}{6} = 10200 \text{ sol. in., Ans.}$$

2. The back of a wedge is 12 inches long by 6 wide; its edge is 15 and its altitude 18; what is its solidity?

Ans. 702 inches.

496. Prob. 28.—To find the solidity of a rectangular prismoid,

Rule.—*To the sum of the areas of the two bases add 4 times the area of a section parallel to and equally distant from the two bases, and this latter sum multiplied by* $\frac{1}{6}$ *the altitude of the prismoid will give the solidity.*

NOTE 1.—This rule will give the contents of *any* prismoid.

NOTE 2.—A *rectangular* prismoid may be divided into 2 wedges by passing a plane through the opposite parallel edges of the upper and lower bases, and then the contents may be found by the rule in Prob. 27.

Let the scholar solve the following examples by each rule.

Ex. 1. The lower base of a rectangular prismoid is 24 by 20 inches, the corresponding sides of the upper base are 18 by 12 inches, and the altitude is 30 inches; what is its solidity?

$24 \times 20 = 480$, lower base; $18 \times 12 = 216$, upper base:
$21 \times 16 \times 4 = 1344$, 4 times the parallel section, $\therefore$
$(480 + 216 + 1344) \times \frac{30}{6} = 10200$ sol. in., Ans. by Prob. 28.

Again, $(24 \times 2 + 18) \times 20 \times \frac{30}{6} = 6600$, larger wedge,
and $(18 \times 2 + 24) \times 12 \times \frac{30}{6} = \underline{3600}$, less wedge,
10200 sol. inches Ans. by Prob. 27.

2. What are the contents of a prismoid whose altitude is 18 feet, lower base 16 by 18 inches, and corresponding sides of the upper base 10 by 12 inches? Ans. 42768 sol. in.

3. What are the contents of a prismoid whose altitude is 18 feet, lower base 16 by 18 inches, and corresponding sides of the upper base 12 by 10 inches? Ans. 42912 sol. in.

How do the 2d and 3d examples differ?

4. What are the contents of a hewn stick of timber that is 40 feet long, 12 by 14 inches at one end, and 8 by 10 inches on the corresponding sides at the other end? Ans. 58240 sol. in.

497. PROB. 29.—To find the solidity of a sphere,

RULE 1.—*Multiply the surface of the sphere by $\frac{1}{3}$ radius.*

NOTE.—This rule is founded on the supposition that a sphere is composed of an infinite number of pyramids whose vertices are at the center of the sphere and whose bases make up the surface of the sphere.

RULE 2.—*Multiply the cube of the diameter by the decimal* .523599; *i. e. by $\frac{1}{6}$ of* 3.141592.

Ex. 1. What is the solidity of a sphere whose radius is 4000 miles? Ans. $268082517333\frac{1}{3}$sol. miles?

2. What is the solidity of a sphere whose diameter is 896000 miles? Ans. $376636634912085333\frac{1}{3}$sol. m.

498. Prob. 30.—To find the solidity of a spherical segment,

Rule.—*Multiply the half sum of the areas of the two bases by the altitude of the segment and to this product add the solidity of a sphere whose diameter is this same altitude.*

Note.—If the segment has but one base the other base is 0 and the same rule applies.

Ex. 1. What is the solidity of a spherical segment whose altitude is 86.60254 inches and the radii of whose bases are 100 and 50 inches? Ans. $2040523.848355309475191954314\frac{2}{3}$sol. in.

2. What is the solidity of a segment of one base, the altitude being 13.39746 feet and the diameter of the base 100 feet?

Ans. 53870.8080150407388347123 52sol. ft.

499. Prob. 31.—To find the solidity of a spherical wedge or ungula,

Rule.—*Find the solidity of the sphere and then say as 360° is to the angle of the wedge, so is the solidity of the sphere to the solidity of the wedge.*

Note.—The angle of the wedge is the same as the angle of the lune that forms its base.

Ex. 1. What is the solidity of a wedge whose angle is 36° in a sphere whose radius is 4000 miles?

Ans. $26808251733\frac{1}{3}$sol. m.

2. What is the solidity of an ungula whose angle is 45° in a sphere whose diameter is 896000 inches?

Ans. $47079579364010666\frac{2}{3}$sol. in.

500. Prob. 32.—To find the solid contents of a spherical pyramid or of a sector,

Rule.—*Having found the area of the triangle, polygon or zone which forms the base, multiply this area by $\frac{1}{3}$ of the radius of the sphere.*

Ex. 1. What is the solidity of a spherical pyramid whose base is a triangle having its angles 80°, 90° and 130°, the radius of the sphere being 4000 miles? Ans. $446804195555\frac{5}{9}$ sol. m.

2. What is the solidity of a pentagonal spherical pyramid in a sphere whose diameter is 5 inches, the angles of the base being 150°, 119° 30′, 75°, 145° and 170° 30′? Ans. $10.9083\frac{1}{18}$ sol. in.

3. What is the solidity of a spherical sector in a sphere whose radius is 12 inches, the arc of the great circle bisecting the sector, or the sectoral angle at the center of the sphere being 120°? Ans. 1809.556992 sol. in.

4. What is the solidity of the remainder of the hemisphere after the sector in Ex. 3 has been taken out?

Ans. 1809.556992 sol. in.

501. Prob. 33. — To find the solid contents of a cube inscribed in a sphere.

Rule. — *Divide the square of the diameter by 3, and the cube of the square root of this quotient will be the solidity sought.*

Note.—The diagonal of the cube is a diameter of the sphere, but the square of the diagonal of any rectangular parallelopipedon is equal to the sum of the squares of its three dimensions; i. e. in the cube, since its three dimensions are equal, the *square of the diagonal is equal to three times the square of either edge;* hence the rule.

Ex. 1. What is the solidity of a cube inscribed in a sphere whose diameter is 10 feet?

$10^2 \div 3 = \frac{100}{3}$; $\sqrt{(\frac{100}{3})^3} = 192.45+$ sol. ft., Ans.

2. What are the solid contents of a cube inscribed in a sphere whose circumference is 18.849552 inches?

Ans. 41.569219+ sol. in.

502. Prob. 34. — To find the contents of a solid of any form,

Rule — *Immerse the solid in a vessel of known form and dimensions partly filled with water, and note the rise of the water in the vessel.*

NOTE. — This rule is founded on the self-evident fact that the volume or bulk of the water displaced is equal to that of the solid immersed.

Ex. 1. An irregular stone, immersed in a cylindrical vessel 10 inches in diameter, raised the water in the vessel 5 inches: what were the contents of the stone? Ans. 392.699 sol. in.

2. There is water 4 inches deep standing in a pail which is 12 inches deep, 10 inches in diameter at the bottom and 13 inches at the top — interior dimensions; what are the contents of a lobster, which, being immersed in this water, will raise it 4 inches? Ans. 415.737341 sol. in.

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§.52. GAUGING.

503. GAUGING is the art of finding the contents of casks or vessels of any form, in gallons, bushels, etc.

504. PROB.—To find the contents of kegs, barrels, etc.

It is difficult or impossible to find the exact contents of kegs, barrels, etc., in consequence of the different curvature of the staves, the difficulty in determining the interior dimensions of the cask, etc.; but, by experience it is found that all such vessels may be gauged with sufficient accuracy by the following

RULE 1.—*Multiply the difference between the bung and head diameters of the cask, by numbers varying from .5 to .7, according as the staves are curved little or much, and add the product to the head diameter to obtain the* MEAN DIAMETER*; then proceed as in finding the contents of a cylinder in Art.* 492.

NOTE 1.—A wine gallon	= 231	cubic inches,	(78, Note 2).
" beer "	= 282	" "	(78, Note 3).
" bushel	= 2150.42	" "	(79, Note 2).

Ex. 1. What are the contents in wine and in beer gallons of a

cask whose length is 44 inches, head diameter 28 inches, and bung diameter 36 inches?

$$(36 - 28) \times .7 = 5.6$$
$$\underline{28.}$$
$$33.6 = \text{mean diameter.}$$

$33.6^2 \times .785398 = $ area of circle (477, Rule 2),

$\therefore \dfrac{33.6^2 \times .785398 \times 44}{231} = $ No. gal. wine, 1st Ans.; but if both numerator and denominator of this fraction are divided by .785398, we shall have

$\dfrac{33.6^2 \times 44}{294}$, very nearly; again,

$\dfrac{33.6^2 \times .785398 \times 44}{282} = $ No. gal. beer, 2d Ans.: but if, as before, both numerator and denominator be divided by .785398, it will give $\dfrac{33.6^2 \times 44}{359}$, very nearly. Hence,

Rule 2.—*Find the mean diameter in inches as in Rule* 1; *then multiply the square of the mean diameter by the length of the cask in inches, and divide the product by* 294 *for wine and by* 359 *for beer gallons.*

2. What are the contents in wine gallons of a cask whose length is 36 inches, and whose head and bung diameters are respectively 16 and 19 inches?

3. What are the contents in beer measure of a cask whose length is 44 inches, and whose head and bung diameters are 26 and 31 inches?

4. What are the contents in bushels of a hogshead whose length is 48 inches, and whose head and bung diameters are 32 and 40 inches?

5. What is the capacity in bushels of a cask whose length is $4\frac{1}{2}$ feet, and whose head and bung diameters are 3 and $3\frac{1}{2}$ feet?

NOTE 2.—To find the contents of vessels in the form of a cylinder, cone, frustum, sphere, etc., proceed as in the Geometrical Problems.

NOTE 3.—To find the contents of irregular vessels or cavities of any description, first fill the vessel or cavity with water, then pour its contents into a vessel of known form and dimensions, and proceed as before.

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§ 53. TONNAGE OF VESSELS.

505. The tonnage of a vessel is the number of tons she will carry, and is determined by measurement.

506. The ship carpenter estimates the tonnage by one rule, and government by another.

CARPENTER'S RULE.—*For a single decked vessel, multiply the length of the keel, breadth at the main beam, and depth of the hold, in feet, together; divide the product by 95, and the quotient is the number of tons.*

For a double decker, take half of the breadth at the main beam for the depth of the hold, and proceed as before.

GOVERNMENT RULE.—*For a single decker, take the length in feet above the deck from the fore part of the main stem to the after part of the stern post, the breadth at the widest part above the main wales on the outside, and the depth from the under side of the deck plank to the ceiling in the hold. From the length take* $\frac{3}{5}$ *of the breadth and the continued product of the remainder, breadth and depth, divided by 95, will give the tonnage.*

For a double decker, take the length above the upper deck; for the depth take half the width and proceed as before.

Ex. 1. What is the carpenter's tonnage of a single decker whose length is 80 feet, breadth 21 feet, and depth 18 feet?

Ans. $318\frac{6}{19}$ tons.

2. What is the carpenter's tonnage for a double decked vessel whose length is 200 feet, and breadth 38 feet?

Ans. 1520 tons.

3. What is the government tonnage of a single decked vessel, whose length is 100 feet, breadth 25 feet, and depth 20 feet?

Ans. $447\frac{7}{19}$ tons.

4. What is the government tonnage of a double decker whose length is 300 feet and breadth 40 feet?

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§ 54. PHILOSOPHICAL PROBLEMS.

Gravity.

507. *Gravity* is the tendency of all bodies to fall towards the center of the earth.

508. The *center of gravity* of a body is "the point about which all the parts of a body exactly balance each other, so that when that point is supported, the whole body is supported."

In a body of uniform density the center of gravity is in the center of the volume or bulk.

The *weight* of a body *is the measure of its gravity.*

509. *Specific gravity* is the weight of a body *compared* with the weight of an equal bulk of some other body *taken as a standard.*

The standard for solids and liquids is *distilled water.*

A cubic foot of distilled water, by statute, weighs 1000oz. $=$ $62\frac{1}{2}$ lbs. avoirdupois.

The *specific* gravity of the standard is 1.

510. If a body is lighter than the standard, its specific grav

ity is less than 1; if heavier, more than 1; thus, if a cubic foot of cork weighs 250oz. ($= \frac{1}{4}$ of 1000oz.), its specific gravity is $\frac{1}{4}$; and if a cubic foot of zinc weighs 7000oz. ($=7$ times 1000 oz.), its specific gravity is 7.

511. PROB. 1.—To find the specific gravity of a solid,

RULE.—*Divide the weight of the body by the loss of weight it sustains when it is immersed in water; i. e. divide its true weight by the weight of an equal bulk of distilled water.*

NOTE.—If the body is lighter than water, as e. g. cork, it must be attached to some heavier body in order to immerse it; and in that case it loses *all* its own weight, together with the amount it diminishes the weight of the body attached to it for the purpose of sinking it.

Ex. 1. A piece of copper weighs 1668.75 lbs. in air, and 1481.25 lbs. in water: what is its specific gravity?

Ans. 8.9.

2. A piece of cork weighs 100oz.; but, being sunk in water, by attaching it to a piece of iron previously balanced in water, it requires 300oz. less to balance the iron; what is the specific gravity of the cork? Ans. $\frac{1}{4}$.

512. PROB. 2.—To find the specific gravity of a liquid,

RULE.—1. *Weigh a solid in the air, then in water, and then in the given liquid.*

2. *Divide the loss of weight in the liquid whose specific gravity is sought by its loss of weight in water; i. e. divide the weight of the liquid by the weight of an equal bulk of water.*

Ex. 1. A piece of lead weighs $709\frac{3}{8}$ lbs.; when immersed in distilled water it weighs $646\frac{7}{8}$ lbs., and in sea-water its weight is 645 lbs. What is the specific gravity of sea-water?

Ans. 1.03.

2. A piece of iron weighing $486\frac{1}{4}$ lbs., upon being immersed in water weighed $423\frac{3}{4}$ lbs.; and in linseed oil it weighed $427\frac{1}{2}$ lbs.; what is the specific gravity of linseed oil? Ans. .94.

513. The specific gravity of the heaviest known substance,*

	Platina, is	22.	Copper,	8.90
That of	Gold,	19.25	Steel,	7.84
	Quicksilver,	13.58	Iron,	7.78
	Lead,	11.35	Tin,	7.29
	Silver,	10.47	Zinc,	7.
	Precious Gems,	from 3 to 4		
	Minerals,	" 2 " 3		
	Liquids,	" $\frac{3}{4}$ " 2		
	Woods,	" $\frac{1}{4}$ " $1\frac{1}{3}$		

The specific gravity of the lightest known substance,

Hydrogen gas, is about .000083, water being 1.

What is the specific gravity of platina, hydrogen gas being the standard?

What is the specific gravity of hydrogen gas, platina being the standard?

514. Below the earth's surface the gravity of a body varies as the distance from the center of the earth; thus, if a body at the surface weighs 1 lb., then at $\frac{1}{4}$ of the distance from the center to the surface, the same body would weigh $\frac{1}{4}$ lb.; at $\frac{1}{2}$ the distance the body would weigh $\frac{1}{2}$ lb., etc., etc.

Again, above the earth's surface gravity varies inversely as the square of the distance from the earth's center; thus, a body weighing 1 lb. at the surface of the earth, would, at 2 times as great a distance from the center, weigh $\frac{1}{4}$ lb.; at 3 times the distance it would weigh $\frac{1}{9}$ lb., etc., etc.

515. PROB. 3.—The weight of a body at the surface of the earth being given to find its weight at any given distance below the surface,

RULE.—*Make a common fraction by writing the radius of the earth for a denominator, and the distance of the body from the*

* Olmsted.

center of the earth for a numerator, and then multiply the weight of the body at the surface by this fraction.

Ex. 1. A body at the surface of the earth weighs 48 lbs., what would it weigh 1000 miles below the surface, supposing the earth's radius to be 4000 miles? Ans. 36 lbs.

2. A body at the surface of the earth weighs 8000 lb., what is its weight 3000 miles below the surface? What at the center?

1st Ans., 2000 lbs. 2d Ans., 0.

516. PROB. 4.—The weight of a body at the surface of the earth being given, to find its weight at any given distance above the surface,

RULE.—*Make a fraction, writing the radius of the earth for the numerator and the distance of the body from the center of the earth for the denominator, and then multiply the weight of the body at the surface of the earth by the square of this fraction.*

Ex. 1. A body at the surface of the earth weighs 36 lb., what would it weigh 4000 miles above the surface?

$36 \times (\frac{4000}{8000})^2 = 36 \times (\frac{1}{2})^2 = 36 \times \frac{1}{4} = 9.$ Ans. 9 lb.

2. If a body weighs 36 lb. at the surface, what will it weigh 8000 miles above the surface? Ans. 4 lb.

3. If a stone weighs 1 ton at the surface of the sea, what will be its weight on the top of a mountain 5 miles high, supposing the diameter of the earth to be 8000 miles?

Ans. 1995 lb. +.

517. By observation and experiment it is found that, near the surface of the earth, a body will fall $16\frac{1}{12}$ feet from a state of rest in one second of time, and, by the laws of gravity, the distances a body will fall in different times will vary as the squares of the times; i. e., if a body fall $16\frac{1}{12}$ feet in 1 second, then in 2, 3 or 4 seconds it will fall 4, 9 or 16 times $16\frac{1}{12}$ feet, etc., etc., $\therefore$,

518. PROB. 5.—To find how far a body will fall in any given time,

RULE.—*As* 1 *to the square of the time in seconds, so is* $16\frac{1}{12}$ *feet to the number of feet the body will fall in the given time.*

NOTE 1.—This rule is true if the body falls in a *vacuum*, but if it falls through the air the resistance is very great when the velocity is great and the result obtained according to the rule is much too large.

Ex. 1. How far will a body fall from a state of rest in 10 seconds? Ans. $1608\frac{1}{3}$ feet.

2. Suppose a body to have been falling 20 seconds, how far has it fallen during the last 10 seconds? Ans. 4825 feet.

NOTE 2.—Many other problems upon falling bodies might be given, but the discussion of the principles relating to them, appropriately belongs to the higher mathematics.

MECHANICAL POWERS.

519. That which communicates or tends to communicate motion to a body, is called a *force* or *power*.

The body which receives motion, or on which a force is expended, is called a *weight*. Force may be applied to a weight by the aid of a *lever*, *wheel and axle*, *pulley*, *inclined plane*, *screw* or *wedge*, and these instruments, six in number, are called the *mechanical powers*.

THE LEVER.

520. The *lever* is an inflexible bar or rod, movable about a fixed point; this point is called the *fulcrum* or *prop*.

521. Levers are of *three* kinds.

1st. Where the fulcrum is between the power and weight.
2d. Where the weight is between the fulcrum and power.
3d. Where the power is between the fulcrum and weight.

522. In the use of either of the three kinds of lever, an equilibrium will be produced when the power is to the weight as the distance of the weight from the fulcrum to the distance of the power from the fulcrum.

(a) Levers of the 1st kind.

Ex. 1. What weight may be sustained on the end of a lever

10 feet long by a force of 100 lb. at the other end, the prop being 2 feet from the weight? Ans. 400 lb.

2. What weight may be sustained by a force of 25 lb. on a lever of 6 feet, the prop being 6 inches from the weight?

3. What power will be required to balance a weight of 2000 lb. on a lever 12 feet in length, the fulcrum being 1 foot and 6 inches from the weight. Ans. $285\frac{5}{7}$ lb.

4. What power is required to sustain 50 lb. on a lever of 3 feet, the fulcrum to be 5 inches from the weight?

5. A power of 200 lb. is applied to a lever 12 feet from the fulcrum; at what distance from the fulcrum will a weight of 4000 lb. be balanced? Ans. $7\frac{1}{5}$ inches.

6. How far from the weight must the fulcrum be placed in order that a power of 10 lb. may balance a weight of 75 lb. on a lever 4 feet long?

7. A weight of 12000 lb. is attached to a lever 2 feet from the prop; at what distance from the prop must a power of 300 lb. be applied to balance the weight? Ans. 80 feet.

8. At what distance from the prop must a power of 40 lb. be placed to balance 550 lb. 9 inches from the prop?

(b) Levers of the 2d kind.

9. A force of 200 lb. is applied to a lever of the 2d kind 8 feet from the fulcrum; what weight will be sustained 6 inches from the fulcrum? Ans. 3200 lb.

10. What power applied 8 feet from the fulcrum will be sufficient to balance a weight of 8000 lb. 10 inches from the fulcrum? Ans. $833\frac{1}{3}$ lb.

11. At what distance from the fulcrum must a power of 50 lb. be applied to balance a weight of 750 lb. 2 feet from the fulcrum? Ans. 30 feet.

12. At what distance from the fulcrum may a weight of 3000 lb. be sustained by a power of 150 lb. applied 6 feet from the fulcrum? Ans. $3\frac{3}{5}$ inches.

(c) Levers of the 3d kind.

13. A force of 100 lb. is applied to a lever of the 3d kind at the distance of 5 feet from the fulcrum; what weight will be balanced 20 feet from the fulcrum? Ans. 25 lb.

14. What power applied 6 feet from the fulcrum will be sufficient to sustain 50 lb. 18 feet from the fulcrum?

Ans. 150 lb.

15. At what distance from the fulcrum must a power of 300 lb. be applied to balance 60 lb. 30 feet from the fulcrum?

Ans. 6 feet.

16. At what distance from the fulcrum may a weight of 100 lb. be sustained by a force of 500 lb. applied 4 feet from the fulcrum? Ans. 20 feet.

NOTE.—In the use of a lever of the 1st kind there may be a *mechanical advantage* or *disadvantage* (i. e., the weight may be greater or less than the power) according as the power or weight is farthest from the fulcrum. In using one of the 2d kind *there must be a mechanical advantage*, and in the 3d kind *a disadvantage*. The principles of a lever of the 3d kind are usually exemplified in raising a long ladder or pole from a horizontal to a vertical position.

THE WHEEL AND AXLE.

523. In using the wheel and axle as a mechanical power they are firmly attached to each other and turn together, the power being usually applied to the circumference of the wheel, by means of a rope, and the weight to the axle, on the opposite side.

524. An equilibrium is produced when the power is to the weight as the radius of the axle to the radius of the wheel. The principle is the same as in the lever, the radius of the axle corresponding to the shorter arm and that of the wheel to the longer arm.

Ex. 1. If the radius of the wheel is 3 feet and the radius of the axle 2 inches, what weight may be sustained on the axle by a force of 50 lb. applied to the wheel? Ans. 900 lb.

2. The radius of the wheel being 3 feet and that of the axle

2 inches, what power applied to the wheel will balance 900 lb. on the axle?

3. What must be the radius of the axle if a power of 50 lb. on a wheel of 3 feet radius balances 900 lb. on the axle?

4. What is the radius of a wheel on which a power of 50 lb. balances 900 lb. on an axle of 2 inches radius?

5. What is the circumference of a wheel on which a power of 12 lb. balances 160 lb. on an axle whose radius is 3 inches?

Ans. 20 feet 11.32736 inches.

THE PULLEY.

525. A pulley is a small wheel, movable about an axis by means of a cord or rope passing over the wheel. The axis may be stationary or movable, i. e. susceptible of rising or falling. If the axis is stationary the pulley is called a *fixed pulley;* if movable the pulley is said to be *movable.*

526. In the use of the fixed pulley there is no mechanical advantage, but it is very convenient in changing the direction of the power; i. e. in enabling us to apply a power in one direction, and thereby to move a weight in some other direction.

527. Blocks, i. e. combinations of 2, 3, or more pulleys, are often used.

528. In the use of pulleys an equilibrium is produced when the power is to the weight as 1 to the number of ropes; i. e. when power : weight : : 1 : twice the number of movable pulleys.

NOTE.—There is really but 1 rope used for a block of pulleys, though it is customary to consider the number of ropes twice as great as the number of movable pulleys, the *parts* of the rope on opposite sides of the pulley *being called different ropes.*

Ex. 1. What power must be applied to a rope passing around one movable pulley to balance a weight of 600 lb.?

Ans. 300 lb.

2. In a block of 4 movable pulleys what weight will be sustained by a power of 200 lb.?

THE INCLINED PLANE.

529. An inclined plane is a plane which is oblique to the horizon -- neither horizontal nor vertical.

530. In the use of the inclined plane, when the power is applied in a line parallel to the length of the plane, an equilibrium is produced when the power is to the weight as the hight of the plane to its length.

Ex. 1. An inclined plane is 40 feet long and 5 feet high; what weight will be balanced by a power of 200 lb.?

Ans. 1600 lb.

2. An inclined plane is 5 feet high and 40 feet long; what power is requisite to balance a weight of 1600 lb.?

3. A weight of 1600 lb. is balanced on an inclined plane whose length is 40 feet, by a power of 200 lb.; what is the hight of the plane?

4. What is the length of an inclined plane whose hight is 5 feet, and on which a power of 200 lb. sustains a weight of 1600 lb.?

(a) When the power is applied in a line parallel to the base of the plane, an equilibrium is produced when the power is to the weight as the hight of the plane to the length of its base.

5. What weight will be sustained on an inclined plane whose hight is 4 feet, and the length of whose base is 12 feet, by a power of 16 lb. applied in a line parallel to the base?

Ans. 48 lb.

6. What power will balance a weight of 48 lb. on a plane whose hight is 4 feet and base 12 feet, the power acting in a line parallel to the base?

7. A power of 16 lb., acting in a line parallel to the base of a

plane whose hight is 4 feet, balances 48 lb.; what is the length of the base?

8. A power of 16 lb. balances a weight of 48 lb. on a plane the length of whose base is 12 feet, the power acting in a line parallel to the base; what is the hight of the plane?

The Screw.

531. A screw is a cylinder having a thread coiled spirally around it, in such a manner that the thread is equally inclined to the base of the cylinder throughout its entire length.

532. The principle of the screw is the same as that of the inclined plane. The length of the thread in going once around the cylinder is the length of the plane; the circumference of the cylinder is the base of the plane; and the distance between two contiguous threads in a line parallel to the axis of the cylinder is the hight of the plane.

The power is seldom applied directly to the screw, but usually at the end of a lever which enters a mortise in the screw, and in a line parallel to the base of the plane.

533. In the use of the screw an equilibrium is produced, when the power is to the weight as the distance between two contiguous threads in a line parallel to the axis of the screw, to the circumference of the circle made by one revolution of the power.

Ex. 1. What pressure may be exerted at the head of a screw by applying a power of 200 lb. at the end of a lever 10 feet long, the threads of the screw being 1 inch apart?

Ans. 150796.416 lb.

2. The pressure exerted at the head of a screw is 150796.416 lb., the threads of the screw are 1 inch apart, and the power is applied at the end of a lever 10 feet long; what is the power?

3. A power of 200 lb. exerts a pressure of 150796.416 lb. at

the head of a screw whose threads are 1 inch apart; what is the length of the lever to which the power is applied?

4. A power of 200 lb. applied at the end of a lever 10 feet long, exerts a pressure of 150796.416 lb. at the head of a screw; what is the distance between the threads of the screw?

The Wedge.

534. For definition of the wedge, see Art. 462, Def. 51.

Wedges "are made of such variety of shapes, and forces are applied in such various ways, that, of all the mechanical powers, the wedge is that whose properties are least capable of being brought to mathematical calculation."*

Only the simplest case will be presented here, viz., when the sides of the wedge are equal rectangles, i. e. when the wedge is isosceles, and also when the power acts perpendicularly on the center of the back, and the weight or resistances act at right angles to the sides.

535. In this case the power is to the weight, i. e. the sum of the resistances on the two sides, as the thickness of the back to the sum of the lengths of the two sides; i. e. the power is to the weight as $\frac{1}{2}$ the thickness of the back to the length of one side.

Ex. 1. The back of an isosceles wedge is 4 inches thick, and the length of one side is 12 inches; what pressure will be exerted on one side, by a force of 500 lb. applied to the back?

Ans. 1500 lb.

2. A wedge whose back is 4 inches thick, and sides 12 inches long, has a pressure of 3000 lb. upon each side; what force applied to the back of the wedge will overcome the resistance?

3. Upon the back of a wedge 4 inches thick, the application of a force of 500 lb. exerts a pressure of 3000 lb. upon each side; what is the length of a side?

4. The length of one side of a wedge is 12 inches, and a force

* Olmsted.

of 500 lb. upon the back will exert a pressure of 3000 lb. upon each side; what is the thickness of the back?

5. The length of one side of a wedge is 15 inches, and the edge forms an angle of 60°; what pressure will be exerted upon each side by a force of 100 lb. applied to the back?

Ans. 100 lb.

Heat: $C = \frac{5}{9}(F - 32)$

$F = \frac{9}{5}(C + 32)$ $F = (C \times \frac{9}{5} + 32)$

STRENGTH OF MATERIALS.

536. The strength of a beam can be determined only by experiment.

Force may be applied to produce fracture in four different ways:—

1st. Laterally; i. e. by pressing upon the side of the beam when it rests horizontally upon its two ends.

2d. Longitudinally; i. e. to *pull* it in two in the direction of the length.

3d. By torsion; i. e. by twisting it off.

4th. By crushing it, as e. g., when a post is set up to support a very heavy weight.

537. The first of these modes of trying the strength of a beam is the only one considered here, and this in only a few of its most simple applications.

538. The data upon which architects rely in calculating the strength of materials, the results of experiment and Geometrical reasoning, are mainly the following:—

1st. The strength of an oak stick an inch square and a foot long, is estimated at 600 lb., the stick lying horizontally, supported at its ends, and the weight resting upon the central point.

The strength of a bar of iron of the same size is estimated at 2190lb.

2d. The strength of *similar beams* varies as the *square* of their corresponding dimensions; thus, if a stick 1 inch square and 1 foot long will sustain 600lb., then one 2 inches square and 2 feet long will sustain 2400lb.; one 3 inches square and 3 feet long will sustain 5400lb.; i.e. a *similar* stick whose dimensions are twice as great will sustain 600lb. $\times\ 2^2 = 600$lb. $\times\ 4 =$ 2400lb.; one whose dimensions are 3 times as great will sustain 600lb. $\times\ 3^2 = 600 \times 9 = 5400$lb., etc., etc.

3d. The strength of a beam lying on its different sides varies as the depth of its center of gravity below the surface; thus, if a beam of given length, 10 inches wide and 2 inches thick, will sustain 100lb., when lying on its edge, then it will sustain only 20lb., i.e. $\frac{1}{5}$ of 100lb., when resting on its broad side.

4th. The strengths of different beams of the same lengths vary as the area of their transverse sections, multiplied respectively into the depths of their centers of gravity below the surface; thus, if 2 beams, A and B, of the same length, are, A 8 inches wide and 6 inches thick and B 10 inches wide and 2 inches thick, then, if they rest each upon its edge,

A's strength : B's strength :: $8 \times 6 \times 4 : 10 \times 2 \times 5 :: 192 : 100$;
if both rest upon their broad sides, then,
A's strength : B's strength :: $8 \times 6 \times 3 : 10 \times 2 \times 1 :: 144 : 20$;
if A rests upon its edge and B upon its broad side, then,
A's strength : B's strength :: $8 \times 6 \times 4 : 10 \times 2 \times 1 :: 192 : 20$;
if A rests upon its broad side and B upon its edge, then,
A's strength : B's strength :: $8 \times 6 \times 3 : 10 \times 2 \times 5 :: 144 : 100$;
and so of beams of other dimensions.

5th. The strengths of beams of different dimensions vary as the areas of their transverse sections multiplied by the depths of their centers of gravity and divided by the lengths of the beams respectively; thus, suppose a beam, A, is 10 feet long, 10 inches wide and 4 inches thick, and that another beam, B, is 12 feet

long, 8 inches wide and 6 inches thick, then, when the beams rest on their edges,

$$\text{A's strength} : \text{B's strength} :: \frac{10 \times 4 \times 5}{10} : \frac{8 \times 6 \times 4}{12} :: 20 : 16;$$

i. e. if A will sustain 20 tons, B will sustain 16 tons. Again, if they rest upon their broad sides, then

$$\text{A's strength} : \text{B's strength} :: \frac{10 \times 4 \times 2}{10} : \frac{8 \times 6 \times 3}{12} :: 8 : 12;$$

etc. etc.

6th. The tendency to fracture a beam is greatest when the weight rests on the central point, and it varies according to the product of the lengths of the two parts of the beam, measured from the point where the weight rests; thus, if a beam is 16 feet long, the tendency to fracture, when the weight rests on the center or 1, 2, 3, etc., feet from the center, may be represented by the products $8 \times 8 = 64$, $9 \times 7 = 63$, $10 \times 6 = 60$, $11 \times 5 = 55$, etc., etc.

7th. The strength of a beam supported at both ends, and having the weight rest on the middle, is 8 times as great as it is when the beam is supported at one end and the weight rests on the other end; or 4 times as great as that of a beam of $\frac{1}{2}$ the length and supported at one end; thus, if a beam 4 feet long, resting on both ends, will support 1600lb. upon its center, then the same beam firmly fixed in a wall at one end will support only 200lb. suspended from the other end; or a beam of the same breadth and depth and only 2 feet long, fixed at one end, will support only 400lb. at the other end.

NOTE.—In these data the weight of the beam itself is not considered, but in practical life it must not be disregarded.

Ex. 1.—What is the strength of an oak stick 4 inches square and 4 feet long? Ans. 9600lb.

2. What is the strength of an iron beam 1 foot square and 12 feet long? Ans. 315360lb.

3. If a stick 12 inches wide, 4 inches thick and 4 feet long, will sustain 86400lb. when resting on its edge, what will it sustain when resting on its side? Ans. 28800lb.

4. If an iron bar 6 inches wide, 3 inches deep and 3 feet long, will sustain 39420lb. when lying on its side, what will it sustain when resting on its edge? Ans. 78840lb.

5. If a stick 6 feet long, 8 inches wide and 3 inches thick, resting on its edge, will sustain 19200lb., then what will a stick of the same material and length, 12 inches wide and 2 inches thick, sustain when lying on its broad side? Ans. 4800lb.

6. If a bar of iron 4 feet long, 6 inches wide and 3 inches thick, will sustain 59130lb., when resting on its edge, what will a bar of the same length, 8 inches wide and 6 inches thick, sustain when resting on its edge? on its side?

1st Ans. 210240lb.; 2d Ans. 157680lb.

7. What is the strength of an oak beam 12 feet long, 8 inches wide and 2 inches thick, when resting on its edge? side?

1st Ans. 6400lb.; 2d Ans. 1600lb.

8. What is the strength of an iron bar 6 feet long, 6 inches wide and 3 inches thick, when resting on its side? edge?

1st Ans. 19710lb.; 2d Ans. 39420lb.

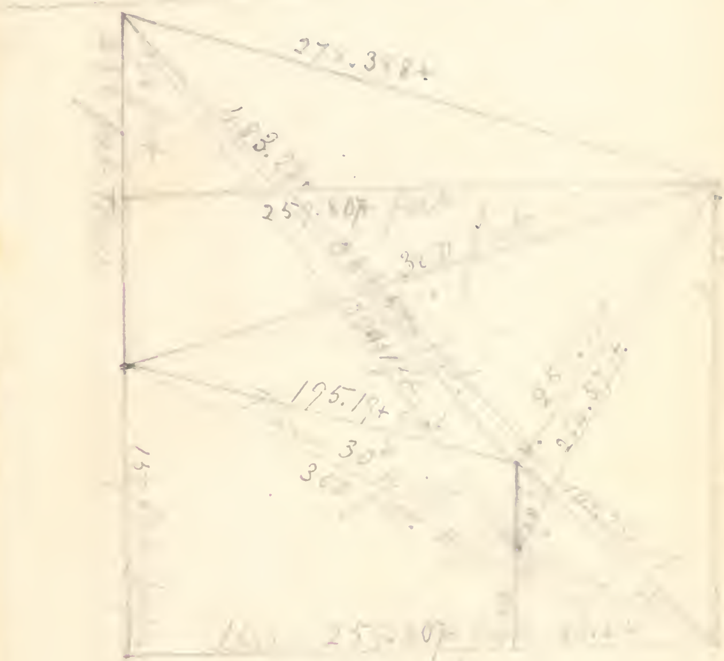
9. What weight will an iron bar 6 feet long and 6 inches square sustain, the weight being placed 2 feet from one end?

Ans. 88695lb.

10. What weight will an oak beam 10 feet long, 10 inches wide and 5 inches thick, sustain 3 feet from its center, the beam resting on its edge? Ans. 46875lb.

11. What weight will an oak beam 4 feet long and 4 inches square sustain, the beam being made fast at one end and the weight applied at the other? Ans. 1200lb.

12. An iron bar 6 inches deep and 3 inches wide is set in masonry so that it extends out of the wall 6 feet; what weight will it sustain at its end? Ans. 4927.5lb.



1st 255.14
2nd 183.
3rd 275.388+
4th 360
5th 362.
6th 259.40